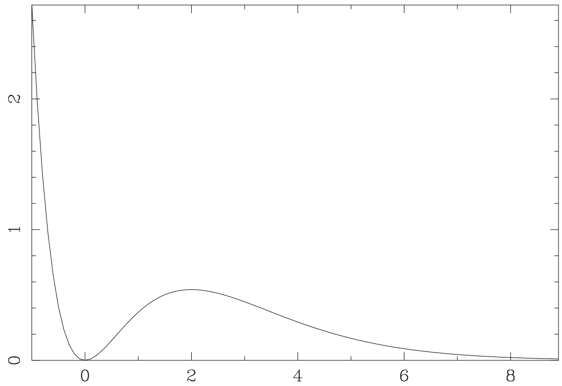
PC10372, Mathematics 2
Example Sheet 1 Solutions

Revision

1) Use the product rule. $\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$. So at the stationary points $2x - x^2 = 0$ giving $x = 0$ or $x = 2$.

Note that as x becomes negative, y tends to ∞. For large postive x, y tends to zero and the only zero is at $x = 0$.



2) $\theta = \tan^{-1}(-2/2) \rightarrow \theta = 3\pi/4$, and $r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2^{3/2}$. Alternatively draw a Argand diagram.

Hence $z^4 = -2 + 2i$, $z^4 = 2^{3/2}\exp\left(i\left(3\pi/4 + 2n\pi\right)\right)$ where $n = 0, 1, 2, 3, ...$ so the solutions are

i) $z = 2^{3/8}e^{i3\pi/16} = 2^{3/8}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right) = 1.0783 + 0.70481i$

ii) $z = 2^{3/8}e^{i11\pi/16} = 2^{3/8}\left(\cos\frac{11\pi}{16} + i\sin\frac{11\pi}{16}\right) = -0.70481 + 1.0783i$

iii) $z = 2^{3/8}e^{i19\pi/16} = 2^{3/8}\left(\cos\frac{19\pi}{16} + i\sin\frac{19\pi}{16}\right) = -1.0783 - 0.70481i$

iv) $z = 2^{3/8}e^{i27\pi/16} = 2^{3/8}\left(\cos\frac{27\pi}{16} + i\sin\frac{27\pi}{16}\right) = 0.70481 - 1.0783i$

3) Have to use L'Hopital's rule as the limit is indeterminate, $0/0$. So differentiate top and bottom

$$\lim_{x\to\pi}\frac{\cos^2(x/2)}{e^x - e^\pi} = \lim_{x\to\pi}\frac{2(1/2)\cos(x/2)\sin(x/2)}{e^x} = \frac{\cos(\pi/2)\sin(\pi/2)}{e^\pi} = 0$$

4)

i)

$$\begin{aligned}
\frac{dy}{dt} &= \cos t - e^{-3t}, \text{ if } y = 3 \text{ when } t = 0 \\
\int dy &= \int \left(\cos t - e^{-3t}\right) dt \\
\therefore y &= \sin t + \frac{1}{3}e^{-3t} + c
\end{aligned}$$

When $t = 0$, $y = 3$, so $3 = 0 + \frac{1}{3} + c$ and hence $c = 8/3$

ii)

$$\begin{aligned}
y\left(1 - x^2\right)^2 \frac{dy}{dx} &= x\left(1 + y^2\right) \\
\text{Rearrange as} & \\
\frac{y}{(1 + y^2)}\frac{dy}{dx} &= \frac{x}{\left(1 - x^2\right)^2} \\
\therefore \int \frac{y}{(1 + y^2)}\, dy &= \int \frac{x}{\left(1 - x^2\right)^2}\, dx
\end{aligned}$$

Now use the substitutions $u = 1 + y^2$ and $w = 1 - x^2$, hence $du = 2y\, dy$ and $dw = -2x\, dx$

$$\begin{aligned}
\therefore \frac{1}{2}\int \frac{du}{u} &= -\frac{1}{2}\int \frac{dw}{w^2} \\
\ln u &= \frac{1}{w} + c_1 \\
\therefore u &= 1 + y^2 = c_2 \exp\left(\frac{1}{w}\right) = c_2 \exp\left(\frac{1}{1 - x^2}\right) \\
\therefore y &= \sqrt{c_2 \exp\left(\frac{1}{1 - x^2}\right) - 1}
\end{aligned}$$

where c_2 is a constant

iii)

$$\begin{aligned}
\frac{dy}{dx} &= x + xy = x(1+y) \\
\int \frac{dy}{1+y} = \int x\,dx & \\
\therefore \ln(1+y) &= \frac{x^2}{2} + c \\
1 + y &= \exp\left(\frac{x^2}{2} + c\right) = A\exp\left(\frac{x^2}{2}\right) \\
y &= A\exp\left(\frac{x^2}{2}\right) - 1
\end{aligned}$$

Since $y = 0$ when $x = 0$, $A = 1$.

5) Take the v terms to one side and the r terms to the other side:

$$\begin{aligned}
\int v\,dv &= -GM\int \frac{1}{r^2}\,dr \\
\frac{v^2}{2} &= \frac{GM}{r} + c
\end{aligned}$$

Using the initial conditions $v = u$ at $r = R$ we get

$$\begin{aligned}
c &= -\frac{MG}{R} + \frac{u^2}{2} \\
\frac{v^2}{2} &= \frac{GM}{r} - \frac{MG}{R} + \frac{u^2}{2} \\
v &= \sqrt{2\left(\frac{GM}{r} - \frac{MG}{R} + \frac{u^2}{2}\right)}
\end{aligned}$$

6) i) The equation is of the form $\frac{dy}{dt}+P(t)=Q(t)$ and the integrating factor is given by

$$\begin{aligned}
I(t) &= \exp\left(\int P(t)\,dt\right) \\
\therefore I(t) &= \exp\left(\int -3\,dt\right) = e^{-3t} \\
\text{Now use } \frac{d(Iy)}{dt} &= QI \\
\therefore \frac{d\left(e^{-3t}y\right)}{dt} &= e^{-2t}e^{-3t} = e^{-5t} \\
\text{Integrating both sides} & \\
e^{-3t}y &= \int e^{-5t}\,dt = -\frac{1}{5}e^{-5t}+c \\
\therefore y &= -\frac{1}{5}e^{-2t}+ce^{3t}. \\
\text{When } t=0, y=1 & \\
\therefore 1 &= -\frac{1}{5}+c, c=\frac{6}{5} \\
\text{So } y=-\frac{1}{5}e^{-2t}+\frac{6}{5}e^{3t} &
\end{aligned}$$

ii) The integrating factor is given by

$$
\begin{aligned}
I(t) = \exp\left(P(t)\,dt\right) &= \exp\left(\int 4\,dt\right) = e^{4t} \\
\therefore \text{ using} \quad & \frac{d\,(Iy)}{dt} = QI \\
\frac{d\left(e^{4t}y\right)}{dt} &= (t-3)e^{4t} \\
\text{Integrating both sides} & \\
e^{4t}y &= \int \left(te^{4t} - 3e^{4t}\right)\,dt \\
\text{Integrate the first term on the RHS by parts using} & \\
u &= t \rightarrow \frac{du}{dt} = 1 \\
\frac{dv}{dt} &= e^{4t} \rightarrow v = \frac{1}{4}e^{4t} \\
\therefore \int te^{4t}\,dt = \frac{1}{4}te^{4t} - \frac{1}{4}\int e^{4t}\,dt &= \frac{1}{4}te^{4t} - \frac{1}{16}e^{4}t + c \\
\text{Hence } e^{4t}y &= \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t} - \frac{3}{4}e^{4t} + c \\
\therefore y &= \frac{1}{4}t - \frac{13}{16} + ce^{-4t} \\
\text{When } t=0, y=1, \text{ hence} \quad & 1 = -\frac{13}{16} + c, \text{ and } c = \frac{29}{16} \\
\therefore y &= \frac{1}{4}t + \frac{29}{16}e^{-4t} - \frac{13}{16}
\end{aligned}
$$

iii) Rewrite as $\frac{dy}{dx} + y\cos x = \cos x$, then the integrating factor is $I = e^{\int \cos x\,dx} = e^{\sin x}$. So

$$
\begin{aligned}
\frac{d}{dx}\left(e^{\sin x}y\right) &= e^{\sin x}\cos x \\
e^{\sin x}y &= \int e^{\sin x}\cos x\,dx \\
\text{Make the substitution } u &= \sin x,\ du = \cos x\,dx \text{ to get} \\
e^{\sin x}y &= \int e^{u}\,du = e^{u} + c = e^{\sin x} + c \\
\therefore y &= 1 + ce^{-\sin x}
\end{aligned}
$$

(Note that the differential equation is also separable.)

7) A homogeneous equation is of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ and can be solved by substituting $y = vx$ which will make the equation separable. Rearrange

$$\begin{aligned}\frac{dy}{dx} &= \frac{y - \sqrt{x^2+y^2}}{x} = \frac{y}{x} - \frac{1}{x}\sqrt{x^2+y^2} \\ &= \frac{y}{x} - \sqrt{\frac{x^2}{x^2} + \frac{y^2}{x^2}} = \frac{y}{x} - \sqrt{1 + \frac{y^2}{x^2}}\end{aligned}$$

So the equation is homogeneous. Substitute $v = y/x$, then $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ (product rule) then

$$\begin{aligned}v + x\frac{dv}{dx} &= v - \sqrt{1+v^2} \\ \to x\frac{dv}{dx} &= -\sqrt{1+v^2} \\ \to \int \frac{dv}{\sqrt{1+v^2}} &= -\int \frac{dx}{x} \\ \to \sinh^{-1} v &= -\ln x + c \text{ (standard integrals)} \\ v &= \frac{y}{x} = \sinh\left(-\ln x + c\right)\end{aligned}$$

Rearranging to the more elegant form

$$y = x\frac{1}{2}\left(e^{-\ln x}e^{c} - e^{-\ln x}e^{-c}\right) = \left(K - \frac{x^2}{K}\right)$$

8) A Bernoulli eqation has the form $\frac{dy}{dx} + g(x)y = h(x)y^n$, so here we have $g(x) = h(x) = 1$ and $n = 3$.

Make the ubstitution $w = y^{1-n} = y^{-2}$, hence $\frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$ and the equation becomes

$$\begin{aligned}-\frac{1}{2}\frac{dw}{dx} + w &= 1 \\ \frac{dw}{dx} &= 2\left(w-1\right), \text{ which is now separable} \\ \int \frac{dw}{(w-1)} &= 2\int dx \\ \therefore \ln|w-1| &= 2x + c \to w - 1 = Ae^{2x} \text{ where } A \text{ and } c \text{ are constants} \\ \therefore w &= y^{-2} = Ae^{2x} + 1 \\ y &= \left(Ae^{2x}+1\right)^{-1/2}\end{aligned}$$

This solution should, as always, be checked by substitution back into the original equation.

9) i) Putting the v terms on one side and integrating we get

$$\begin{aligned} \int \frac{dv}{g+kv} &= -\int dt \\ \frac{1}{k}\ln|g+kv| &= -t+c \\ g+kv &= Ae^{-kt} \to v = -\frac{Ae^{-it}-g}{k} \end{aligned}$$

ii) Re-express the equation as $\frac{dv}{dt}+kv=-g$. The integrating factor is e^{kt} giving

$$\begin{aligned} \frac{d}{dt}\left(e^{kt}v\right) &= -ge^{kt} \\ e^{kt}v &= -\frac{g}{k}e^{kt}+C \\ v &= -\frac{g}{k}+Ce^{-kt} \end{aligned}$$

which is the same as above.

iii) The initial condition gives

$$\begin{aligned} u = -\frac{g}{k}+C \;\; &\to \;\; C = u+\frac{g}{k} \\ \text{Hence } v &= \frac{g}{K}\left(e^{-kt}-1\right)+ue^{-kt} \end{aligned}$$