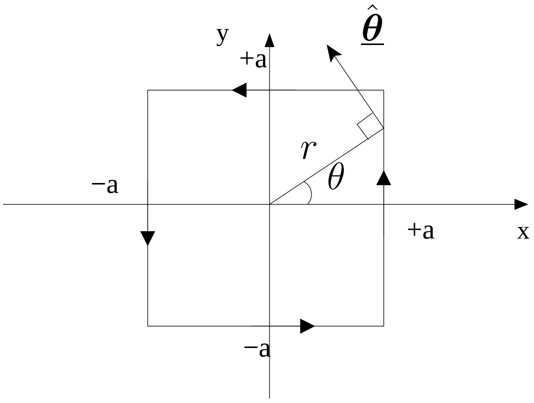
PC10372, Mathematics 2
Example Sheet 11 Solutions

1) Since the system is symmetric, the contributions from all four sides must be equal. Therefore we will only consider the right ($x = a$) edge and multiply the result by 4.



$$\oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}}l = 4\frac{\mu_0 I}{2\pi}\int_{\text{right}} \frac{\hat{\underline{\boldsymbol{\theta}}} \cdot \underline{\mathbf{d}}l}{r}$$

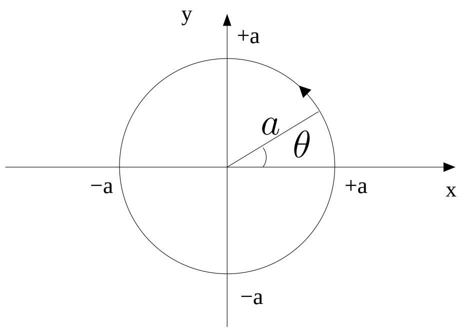
Do the integral in polar coordinates so that we integrate over θ.

$$\begin{aligned}
\hat{\underline{\boldsymbol{\theta}}} &= -\sin\theta\underline{\mathbf{i}} + \cos\theta\underline{\mathbf{j}}, \quad \underline{\mathbf{d}}l = dy\underline{\mathbf{j}} \text{ for right edge} \\
\hat{\underline{\boldsymbol{\theta}}} \cdot \underline{\mathbf{d}}l &= \cos\theta\, dy \\
\text{also } r &= \frac{a}{\cos\theta} \\
\rightarrow \oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}}l &= \frac{2\mu_0 I}{\pi a}\int_{-a}^{a} \cos^2\theta\, dy \\
\text{But } y &= a\tan\theta \rightarrow dy = a\sec^2\theta\, d\theta \\
\rightarrow \oint \underline{\mathbf{B}} \cdot \underline{\mathbf{d}}l &= \frac{2\mu_0 I}{\pi}\int_{-\pi/4}^{\pi/4} d\theta = \mu_0 I
\end{aligned}$$

which of course is just the result from Ampère's law.

2)

$$\oint_C \underline{\mathbf{A}} \cdot \underline{\mathbf{d}}l = \oint_C 5x\underline{\mathbf{j}} \cdot (dx\underline{\mathbf{i}} + dy\underline{\mathbf{j}}) = 5\oint_c x\, dy$$



$x = a\cos\theta$, $y = a\sin\theta$ and $dy = a\cos\theta\, d\theta$

$$\oint_C \underline{\mathbf{A}} \cdot \underline{\mathbf{d}l} \;=\; 5\int_0^{2\pi} d\theta\,(a\cos\theta)(a\cos\theta\, d\theta) = 5a^2\int_0^{2\pi}\cos^2\theta\, d\theta = 5\pi a^2$$

3)

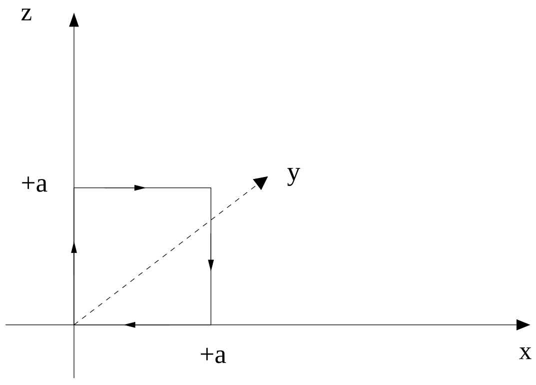
$$\oint_C \underline{\mathbf{A}} \cdot \underline{\mathbf{d}l} \;=\; \int_S (\nabla \times \underline{\mathbf{A}}) \cdot \underline{\mathbf{d}S}$$

where S is the surface bounded by C, so the interior of the circle.

$$\nabla \times \underline{\mathbf{A}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 5x & 0 \end{vmatrix} = 5\underline{\mathbf{k}} \quad \text{and} \quad \underline{\mathbf{d}S} = \underline{\mathbf{k}}\, dS$$

$$\rightarrow \oint_C \underline{\mathbf{A}} \cdot \underline{\mathbf{d}l} \;=\; 5\int_S dS = 5 \times \text{ area of circle} = 5\pi a^2$$

4) Choose clockwise loop, so that the normal to the loop is $\underline{\mathbf{j}}$ (defined by the righthand rule).



$$\oint_C \underline{\mathbf{A}} \cdot \underline{\mathbf{d}l} \;=\; \oint z^2\, dx$$

But $dx = 0$ for the left and right edges while for the bottom edge $z = 0$ so there is only a contribution from the top edge.

$$\oint_c \underline{\mathbf{A}} \cdot \underline{\mathbf{d}l} = \int_{top} z^2\, dx = a^2\int_0^a dx = a^3$$

5)

$$
\begin{aligned}
\nabla \times \underline{\mathbf{A}} &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -y^2 & 0 \end{vmatrix} = 2z\underline{\mathbf{j}} \\
\text{and } \underline{\mathbf{d}S} &= dx\,dz\,\underline{\mathbf{j}} \text{ for the clockwise loop} \\
\oint_C \underline{\mathbf{A}} \cdot \underline{\mathbf{d}l} &= \int_S (\nabla \times \underline{\mathbf{A}}) \cdot \underline{\mathbf{d}S} = \int_0^a dx \int_0^a dz\, 2z = [x]_0^a [z^2]_0^a = a^3
\end{aligned}
$$