PC10372, Mathematics 2
Example Sheet 10 Solutions

1) a) Brute force method: Use $x = r\sin\theta\cos\phi$ then $\underline{\mathbf{G}} = 3a\sin\theta\cos\phi\underline{\mathbf{i}}$ on the surface $r = a$ and $\underline{\mathbf{d}A} = a^2\sin\theta\, d\theta\, d\phi\underline{\hat{\mathbf{r}}}$ using the fact that $r = a$ on the surface. Also $\underline{\mathbf{i}}\cdot\underline{\hat{\mathbf{r}}} = (\underline{\mathbf{i}}\cdot\underline{\mathbf{r}})/r = x/r = \sin\theta\cos\phi$. Therefore

$$\begin{aligned}
\int_S \underline{\mathbf{G}}\cdot\underline{\mathbf{d}A} &= 3a^3\int_0^{2\pi} d\phi\,\cos^2\phi\int_0^{\pi} d\theta\,\sin^3\theta \\
\text{But } \int_0^{2\pi} d\phi\,\cos^2\phi &= \frac{1}{2}\int_0^{2\pi}(1+\cos 2\phi)d\phi = \frac{1}{2}[\phi + \frac{1}{2}\sin 2\phi]_0^{2\pi} = \pi \\
\text{and } \int_0^{\pi} d\theta\,\sin^3\theta &= \int_0^{\pi} d\theta\,\sin\theta(1-\cos^2\theta) = [-\cos\theta + \frac{1}{3}\cos^3\theta]_0^{\pi} = \frac{4}{3} \\
\to \int_S \underline{\mathbf{G}}\cdot\underline{\mathbf{d}A} &= 3a^3\pi\frac{4}{3} = 4\pi a^3
\end{aligned}$$

ii) More elegantly, one can exploit the spherical symmetry. Clearly the fluxes of $x\underline{\mathbf{i}}$, $y\underline{\mathbf{j}}$ and $z\underline{\mathbf{k}}$ out of the sphere are all equal, so the flux of $\underline{\mathbf{G}}' = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$ out of the sphere will be the same as for $\underline{\mathbf{G}}$. On the surface $\underline{\mathbf{G}}' = a\underline{\hat{\mathbf{r}}}$ so $\underline{\mathbf{G}}'\cdot\underline{\mathbf{d}A} = a\,dA$ as $\underline{\mathbf{d}A} = \underline{\hat{\mathbf{r}}}\,dA$ for a sphere ($\underline{\hat{\mathbf{r}}}$ is the normal to the surface).

$$\int_S \underline{\mathbf{G}}'\cdot\underline{\mathbf{d}A} = a\times \text{ surface area of sphere} = 4\pi a^3$$

b) Use divergence theorem. $\nabla\cdot\underline{\mathbf{G}} = 3$.

$$\int_S \underline{\mathbf{G}}\cdot\underline{\mathbf{d}A} = \int_V \nabla\cdot\underline{\mathbf{G}} = 3\int_V dv = 3\times\text{volume of sphere} = 3\times\frac{4}{3}\pi a^3 = 4\pi a^3$$

2) $\nabla\cdot\underline{\mathbf{F}} = y^2 + z^2 + x^2$, so the flux out of the box is

$$\begin{aligned}
\int_S \underline{\mathbf{F}}\cdot\underline{\mathbf{d}A} &= \int_{\text{volume}}\nabla\cdot\underline{\mathbf{F}}\,dV = \int_0^a dx\int_0^b dy\int_0^c dz(x^2+y^2+z^2) \\
&= [\frac{1}{3}x^3]_0^a[y]_0^b[z]_0^c + [\frac{1}{3}y^3]_0^b[x]_0^a[z]_0^c + \frac{1}{3}z^3]_0^c[x]_0^a[y]_0^b \\
&= \frac{1}{3}(a^3bc + b^3ac + c^3ab) = \frac{1}{3}abc(a^2+b^2+c^2)
\end{aligned}$$

3) Use cylindrical polar coordinates.

$$\begin{aligned}
\nabla\cdot\underline{\mathbf{F}} &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial\theta}F_\theta + \frac{\partial}{\partial z}F_z \\
&= \frac{1}{r}\frac{\partial}{\partial r}(r^3) + \frac{\partial}{\partial z}(z^2) = 3r + 2z
\end{aligned}$$

Alternatively note that $r^2\underline{\hat{\mathbf{r}}} = r\underline{\mathbf{r}}$

$$\begin{aligned} \nabla \cdot \underline{\mathbf{F}} &= \nabla \cdot (r\underline{\mathbf{r}}) + 2z \\ \text{and } \nabla \cdot (r\underline{\mathbf{r}}) &= r\nabla \cdot \underline{\mathbf{r}} + \underline{\mathbf{r}} \cdot \nabla r = 2r + \underline{\mathbf{r}} \cdot \underline{\hat{\mathbf{r}}} = 3r \end{aligned}$$

So the flux out of the quarter cylinder is

$$\begin{aligned} \int_S \underline{\mathbf{F}} \cdot \underline{\mathbf{d}A} &= \int_V \nabla \cdot \underline{\mathbf{F}}\, dV = \int_0^R r\, dr \int_0^{\pi/2} d\theta \int_0^h dz\, (3r + 2z) \\ &= 3\left[\frac{r^3}{3}\right]_0^R \frac{\pi}{2} h + 2\left[\frac{r^2}{2}\right]_0^R \frac{\pi}{2}\left[\frac{z^2}{2}\right]_0^h \\ &= \frac{\pi}{2} R^3 h + \frac{\pi}{4} R^2 h^2 \end{aligned}$$

4) Divergence theorem gives

$$\frac{1}{3}\int_S \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = \frac{1}{3}\int_V (\nabla \cdot \underline{\mathbf{r}})\, dV$$

For any volume V enclosed by the surface S. But $\nabla \cdot \underline{\mathbf{r}} = 3$, so

$$\frac{1}{3}\int_S \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = \frac{3}{3}\int_V dV = \text{volume enclosed by surface}$$