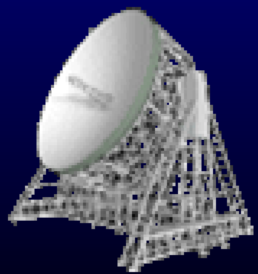
Kepler's Laws and Gravity

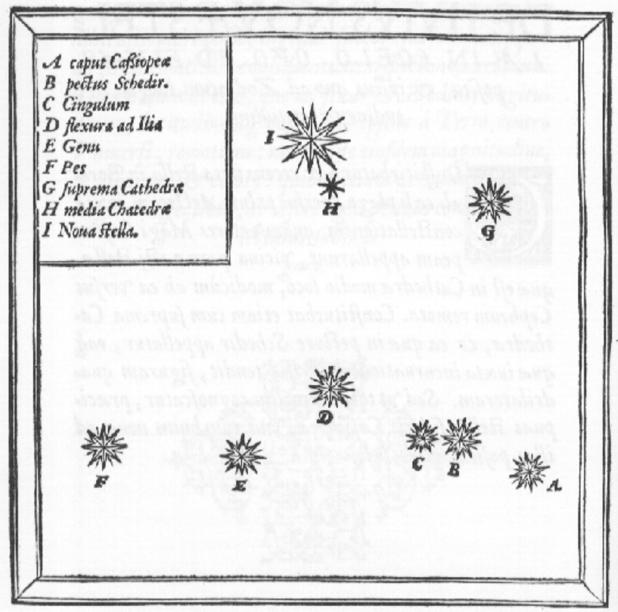
Ian Morison

MANCHESTER
1824
Jodrell Bank Observatory



Observations of a supernova.

- Tycho made detailed observations of the supernova of 1572 – now called Tycho's supernova.
- It was initially brighter than Venus and was visible for 18 months before fading from view.
- Tycho showed with precise observations that its position did not change with respect to the stars close to it in the sky – it was not nearby the Earth.
- This was evidence against the immutable nature of the heavens.



Diſtantiam verò huius ſtellæ à fixis aliquibus in hac Caſsiopeiæ conſtellatione, exquiſito inſtrumento, & omnium minutorum capacj, aliquoties obſeruaui. Inueni autem eam diſtare ab ea, quæ eſt in pectore, Schedir appellata B, 7. partibus & 55. minutis: à ſuperiori verò

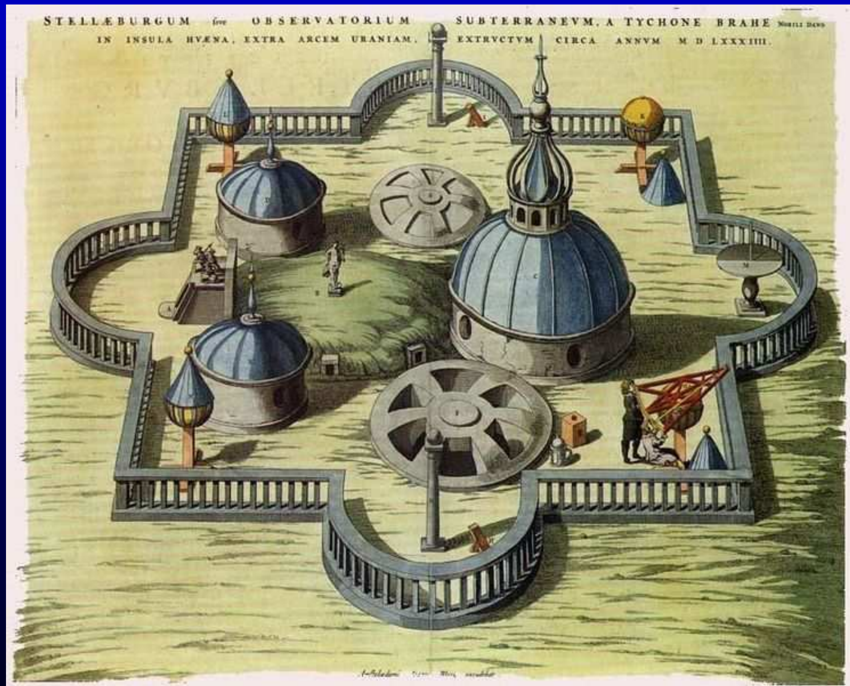
The comet of 1577

- Tycho made accurate observations and showed that the comet must be further than the Moon.
- This contradicted the teaching of Aristotle who had held that they were atmospheric phenomena.
- Furthermore it was very difficult to explain its orbit by uniform circular motion.

Multiple Observations

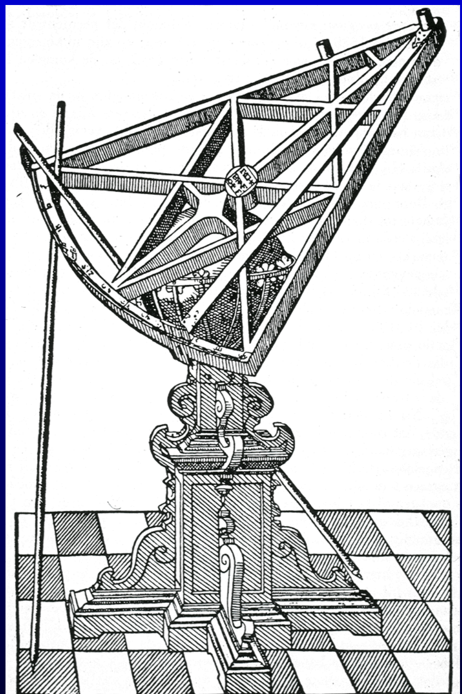
- The several sets of equipment and large staff enabled him to have each observation made 4 times. This, allied to the high quality of the instruments, greatly improved their accuracy.
- Since the time of Ptolemy, the accuracy of positional measurements had remained at ~10 minutes of arc – one sixth of a degree.
- Tycho improved this by a factor of ten to 1 minute of arc!

Tycho Brahe at Uraniborg



Precision Instruments

- He filled Uraniborg with measuring instruments of great precision



But, for 20 years, he also observed the positions of the Planets

- He observed the heavens from ~ 1577 to 1597 and plotted the motion of the planets against the fixed backdrop of the stars

Jupiter and Venus below the Moon

Leaving Hveen and hiring Kepler.

- King Ferdinand died in 1588, and attempts were made in to reign in Tycho’s expenditure.
- He abused his tenants and was summoned to court.
- In 1597 he left Hveen with his family, servants and instruments and wandered around Europe.

Tycho's final years

- In 1599 Tycho was appointed Imperial Mathematician to Emperor Rudolph II.
- He was given a choice of castles (!) and chose Benakty Castle 40 km north of Prague

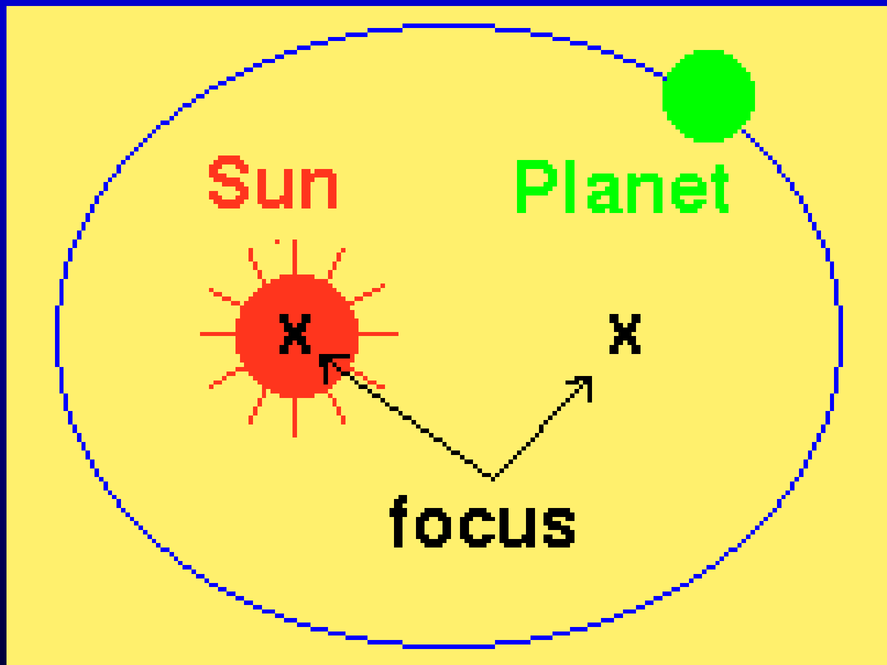
- In 1600 Johannes Kepler came to work with him.
- They worked together for until Tycho died on October 14th 1601 leaving Kepler as his heir.

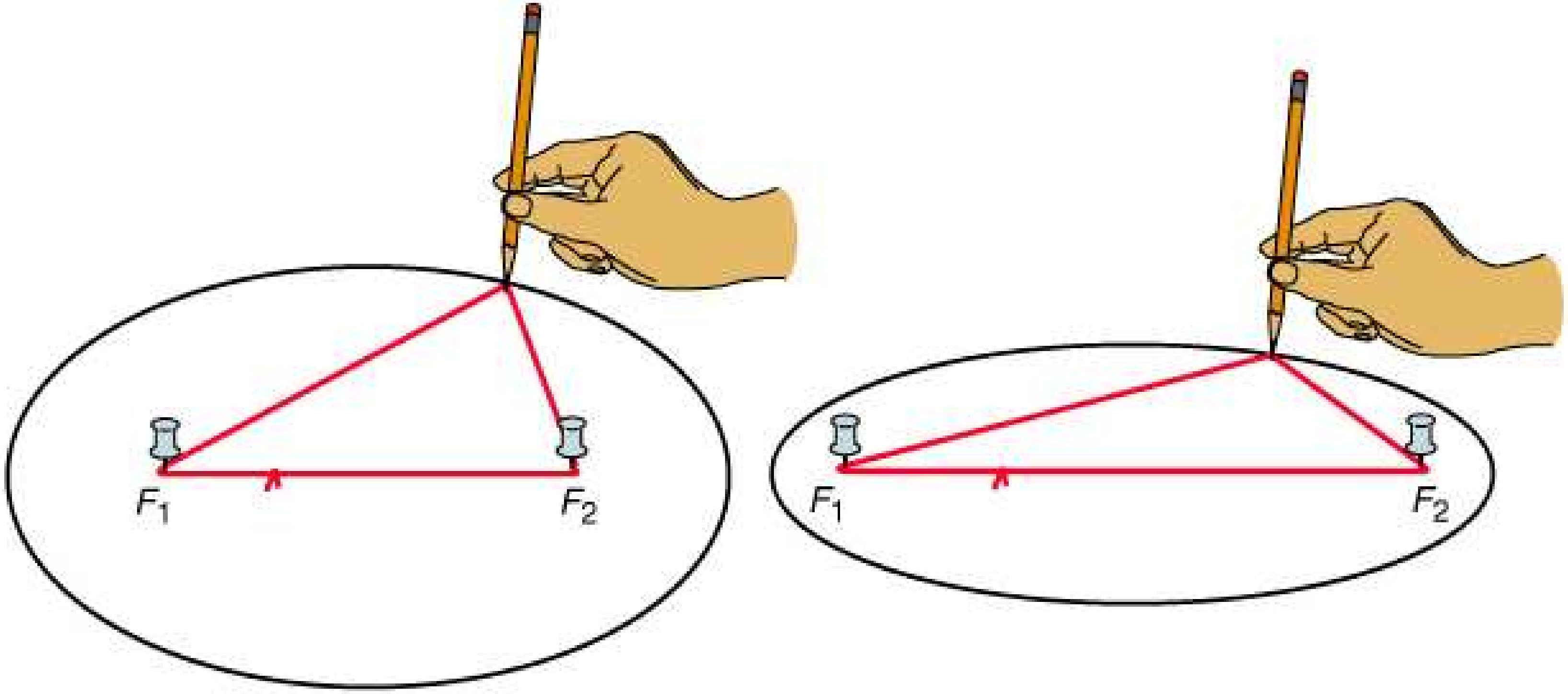
Johannes Kepler

- Born on Dec. 27th 1571 in Weil, Germany.
- He studied at Tuebingen University but supported Copernicus – so was not offered a post there but, instead, became professor of Astronomy in Graz in 1594.
- From the planetary data in Tycho's records, Kepler deduced three empirical laws of planetary motion.

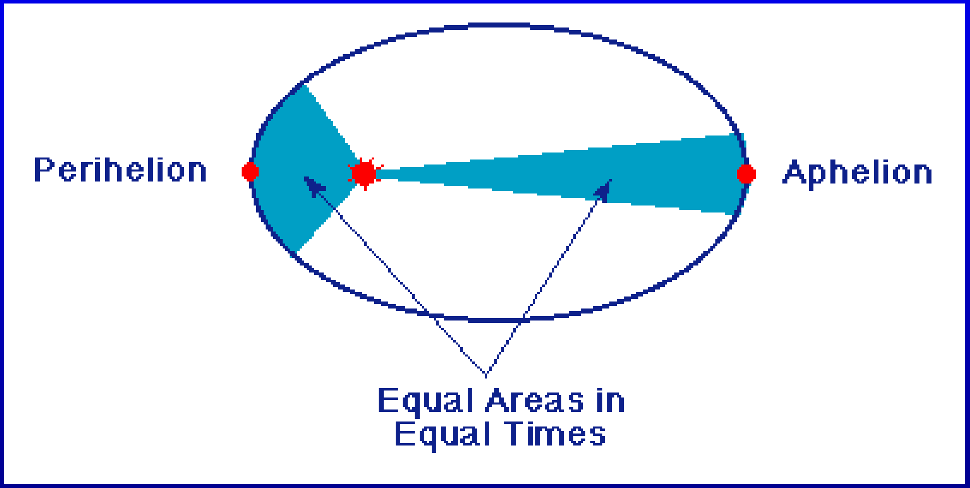
Kepler's three Laws of Planetary Motion.

- 1) The planets move in elliptical orbits with the sun at one focus.

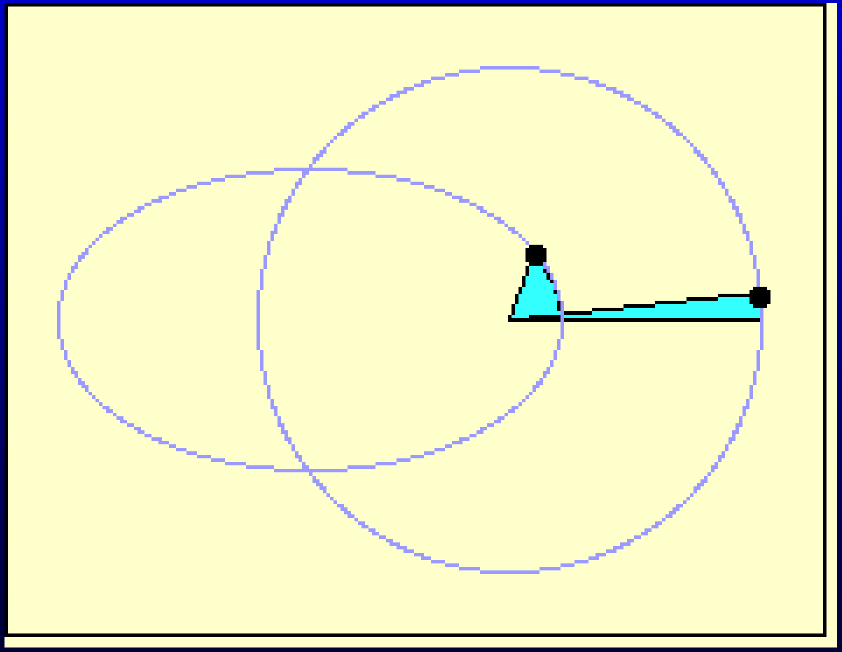



F1
F2
F1
F2

Second Law



- 2) In their orbits around the sun, the planets sweep out equal areas in equal times.



Think about this

- As the planet gets closer to the Sun it loses potential energy.
- But its energy must be conserved.
- So it must gain kinetic energy and so the planet's speed increases as it gets nearer to the Sun.

Third Law

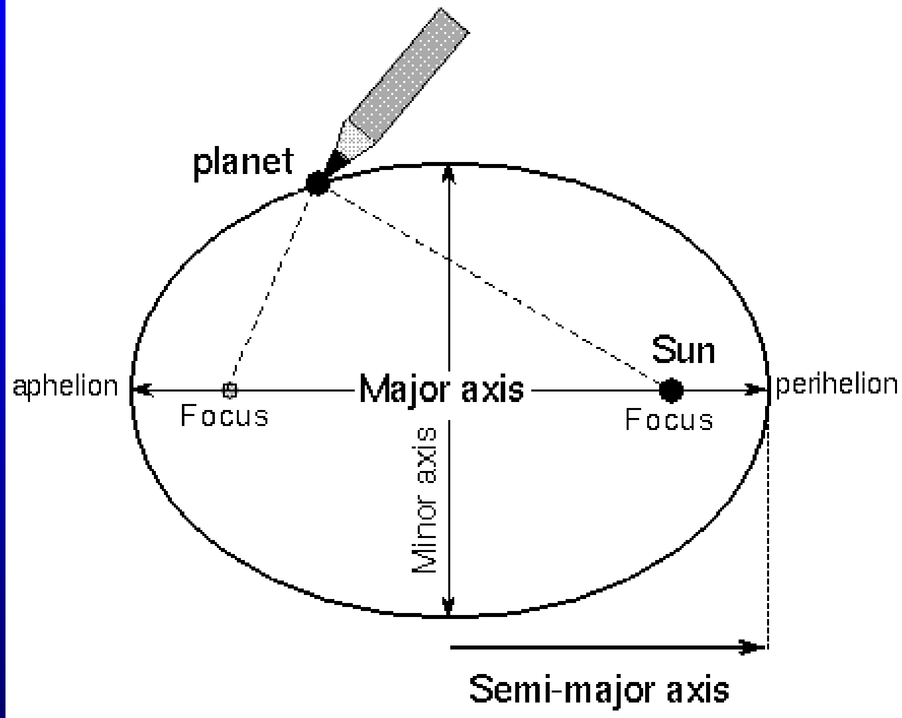
- 3) The squares of the times to complete one orbit are proportional to the cubes of the average distances from the Sun.

$$\frac{P_1^2}{P_2^2} = \frac{R_1^3}{R_2^3}$$

- Remember New York – Times Square

A Complication

- The orbit of a planet is an ellipse so the semi- major axis (a) of the orbit must be used rather than R.



Drawing an **ellipse**: loop string around thumb tacks at each **focus** and stretch string tight with a pencil while moving the pencil around the tacks. The Sun is at one focus.

It should also be noted that Kepler's Third law as stated above is only applicable when one of the bodies is significantly more massive than the other – as is always the case for planets orbiting the Sun.

The value of k, which will be the same for all objects orbiting the Sun, depends on the units chosen. It is conventional to measure the period, T, in units of Earth years and the semi-major axis, a, in units of the Earth's semi-major axis which is termed an Astronomical Unit (AU). In this case $k = 1$.

An example

- What would be the orbital period of a planet whose semi-major axis was 4 times that of the Earth?
- $P^2 = a^3$ (If P is in years and a is in units of the Earth's semi-major axis.)

  $P = a^{3/2}$ = square root of (a) cubed

  = square root of 64 = 8 years

A second Example

The semi-major axis of the dwarf planet Ceres, which orbits the Sun every 4.60 years can be simply found using

$T^2 = k \times a^3$ with $k = 1$, so $a^3 = T^2$ and thus:

$$a = T^{2/3}$$

giving a = 2.76AU.

Kepler's third law can, of course, be applied to any system of planets or satellites orbiting a body. Only the value of the constant of proportionality will be different.

Geostationary Satellites

- Satellite television signals are broadcast from what are termed “Geostationary” orbits above the equator. At a specific distance from the centre of the Earth a satellite will orbit the Earth once per day and so remain in the same position in the sky as seen from a location on the Earth’s surface allowing a fixed reception antenna. How high above the surface of the Earth at the equator would such an orbit be?

- The radius of the Moon’s orbit is 384,400 kilometres and its orbital period around the Earth is 27.32 days.

Using these values we can calculate the constant of proportionality that applies to satellites around the Earth:

$$k = (27.32)^2 / (384{,}400)^3$$
$$= 1.314 \times 10^{-14}$$

For our geostationary satellite, T is 1, so we derive “a” from:

$$1 = k \times a^3$$
$$a = (1/k)^{1/3}$$
$$= 42374 \text{ km}$$

the surface of the Earth is 6400 km from the centre so the satellite is ~ 36,000 km above the surface of the Earth.

The consequences

1) These laws led Newton to the Law of Gravity.

2) The Third Law, with Tycho’s observations, directly enable a superb map of the Solar system to be made to very high precision, BUT they could not give a scale to that map.

The Scale Size of the Solar System

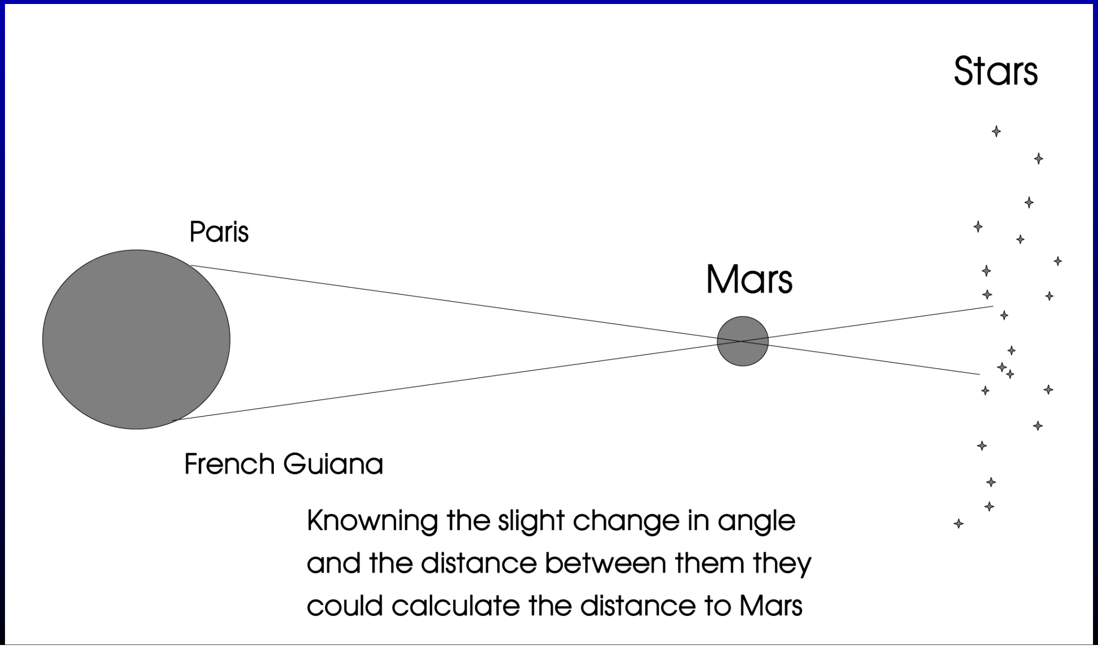
Remember that Kepler's Third Law could provide a superb map, but could not give it a scale.

The Astronomical Unit

- This is the unit with which we measure distances within the Solar System.
- It is the semi-major axis of the Earth's orbit and thus close to the mean distance of the Earth from the Sun.

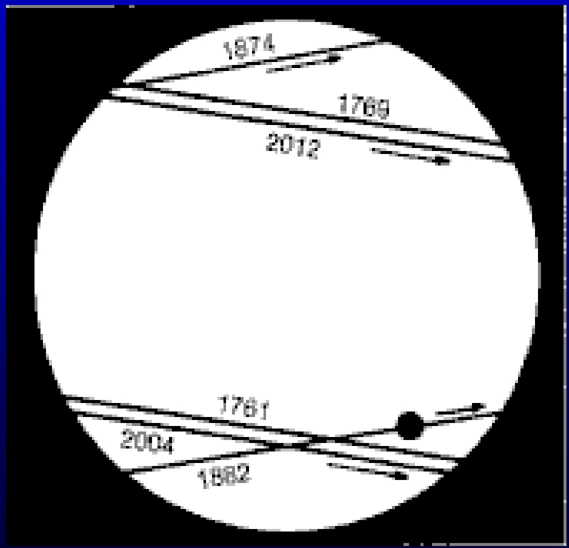
Measurement of the Astronomical Unit

- In 1672 Cassini observed Mars from Paris whilst a colleague observed it from French Guiana in South America. They were thus able to measure its parallax, and hence measure the Earth-Mars distance. Using Kepler's third law they were thus able to calculate the Earth's distance from the Sun.
- He got 140 million km – low but not bad.



Transits of Venus

- Later astronomers observed the transit of Venus (2 per century)
- By timing, from locations all over the Earth, when Venus first entered the Sun's limb and then just before it left, one can measure the parallax of Venus and hence find its distance.



Transits of 1789 and 18th Century

- Value deduced from 1789 transit was between 93 and 97 million miles. (149 to 155 km)
- Enke analysed both 18th century eclipses and deduced 95.25 million miles. (152 km)

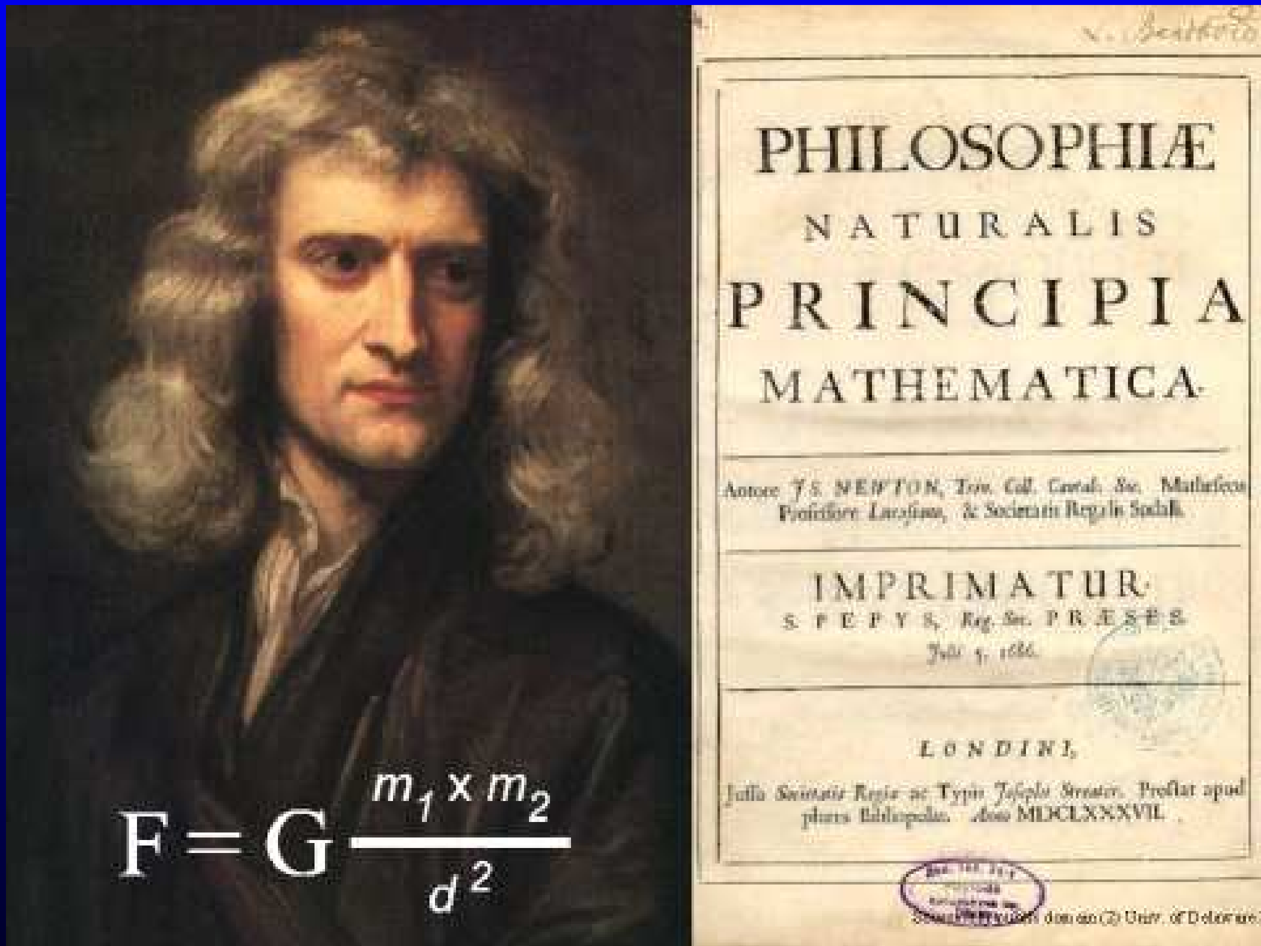
Distance by Radar

- The AU was finally found to high precision by Planetary radars in the US, USSR and UK (using the MK1 Radio-Telescope).
- The result was:
  – 149,597,870.691 km
- So just less than 150 million km or 93 million miles.

The Universal Law of Gravitation

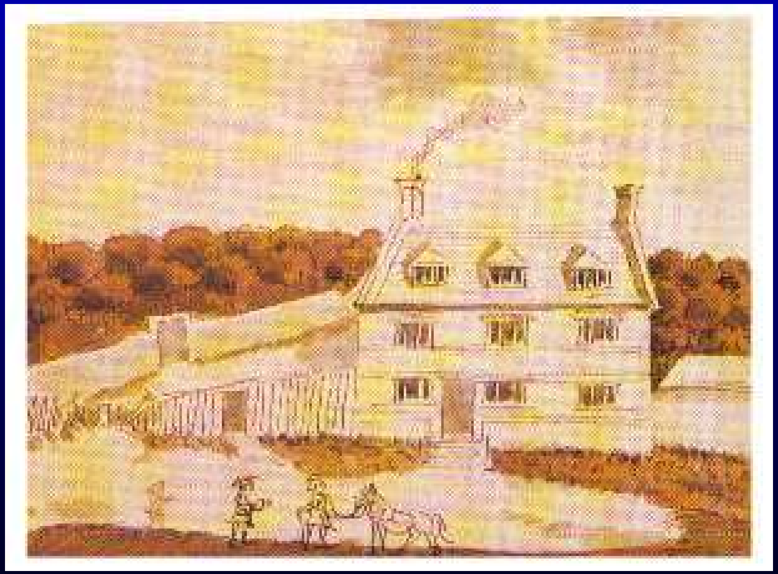
Isaac Newton’s great contribution to Physics

Isaac Newton



- Isaac Newton was born in the manor house of Woolsthorpe, near Grantham in Lincolnshire on Christmas Day 1642.

- He was educated at the Free Grammer School, Grantham and Trinity College Cambridge.

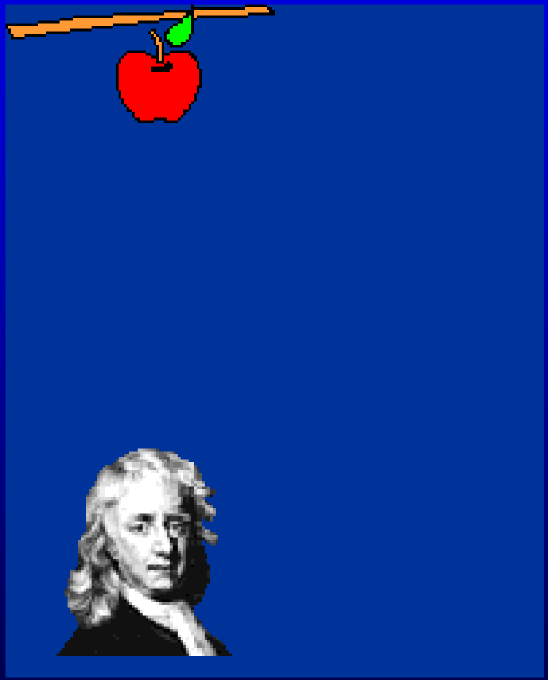


The Plague

- In 1665 and 1666, the Plague affected England. Students at Cambridge were sent home as the University closed. Isaac went to stay with his mother in Woolsthorpe. These next two years were to prove valuable to science, as he spent the time working on his ideas.

The Apple Tree

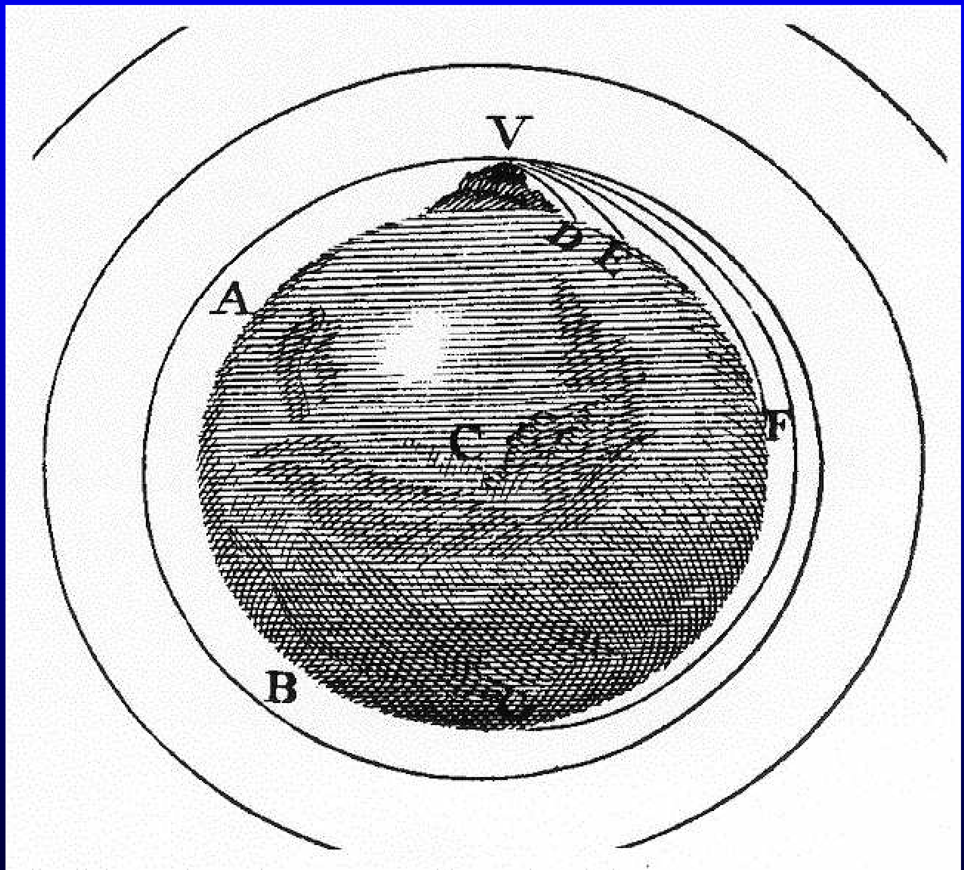
- One story, which is probably untrue, is that Newton was pointed towards his theory of gravity whilst sitting under an apple tree in the orchard. When an apple landed on his head he wondered why the Moon did not fall in the same way.

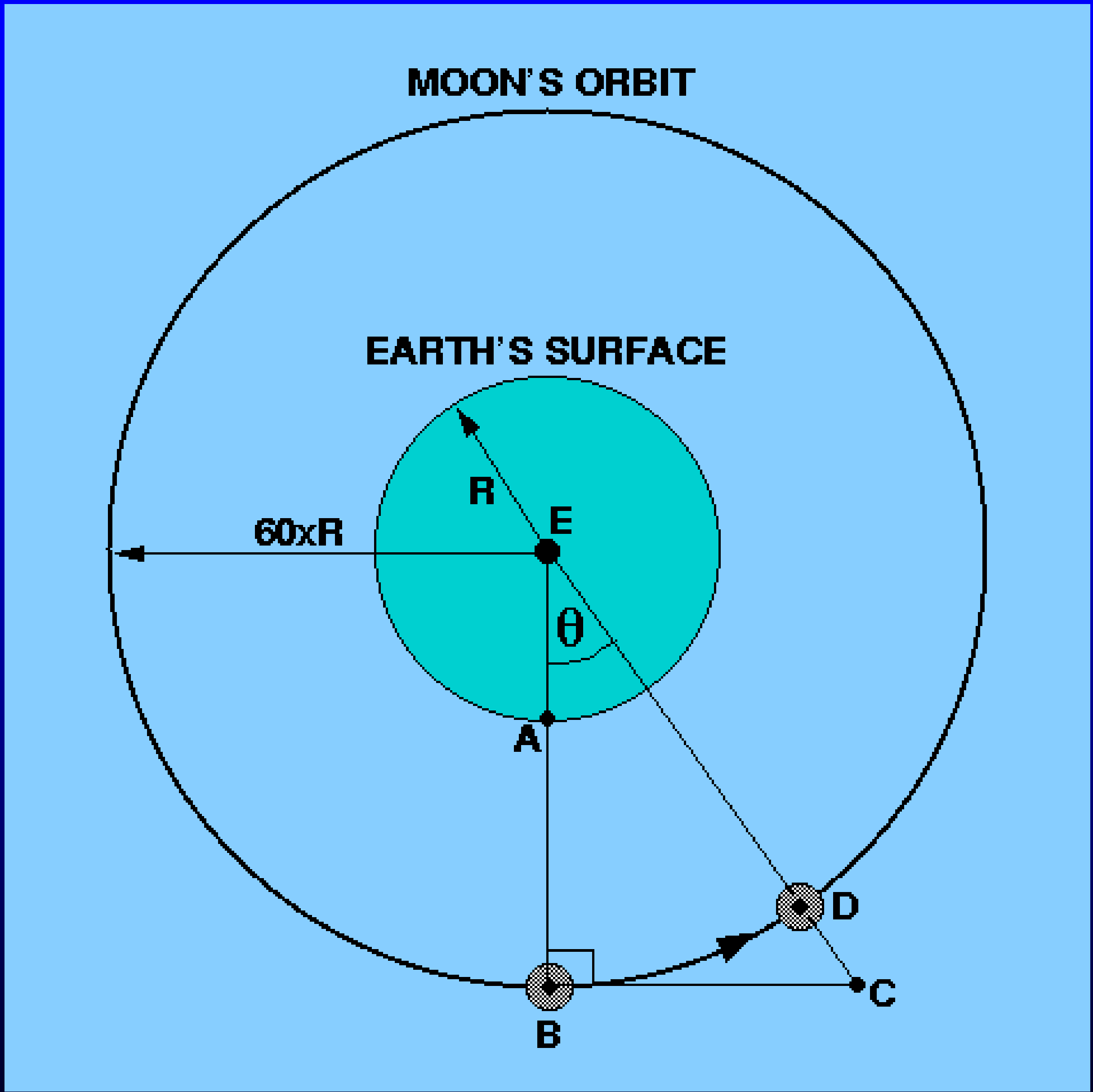


Newton’s moment of Genius

- He realised that the Moon WAS falling – but at just the right rate to keep it at a fixed distance from the Earth.
- Newton began to think of the Earth's gravity as extending out to the Moon's orbit.

Nice Analogy




MOON'S ORBIT
EARTH'S SURFACE
R
E
60xR
θ
A
D
C
B

- He knew that the Moon was ~60 times further away from the centre of the Earth than the surface.
- He calculated the value of g that would have to exist at the distance of the Moon in order for the Moon to "fall" by just the right amount to keep it at a fixed distance from the Surface.
- This was ~1/3600 of the value of g at the surface of the Earth
- So the "strength" of gravity would have to fall as the square of the distance!
  - The inverse square law.

He was able to derive the Law of Gravitation from his laws of motion coupled with Kepler's third law of planetary motion

From his second Law:

Force = mass x acceleration

and the fact that the centripetal acceleration, a, of a body moving at speed v in a circle of radius r is given by v^2/r, he inferred that the force on a mass m in a circular orbit must be given by

$$F = ma = mv^2/\mathrm{r}.$$

- If P is the orbital period of the body, then its speed is

  $v = 2 \pi r / P$

  and Kepler's third law relates the period and the orbital radius by

  $P^2 = kr^3$

  where k is the constant of proportionality. Substituting these last two equations into his second law, Newton found

  $F = 4 \pi^2 m / kr^2$

- that is, the force on the body is proportional to its mass and inversely proportional to the square of its distance from the central body.
- According to Newton's third law, the body at the centre of the orbit feels an equal but opposite force, which must be proportional to the mass of the central body.

Redefining the constant of proportionality in the above equation (which effectively included the mass of the central body), Newton hence obtained his universal law of gravitation

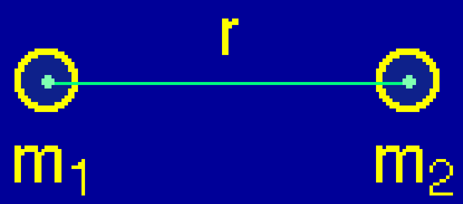
$F = Gm_1m_2 / r^2$,

where F is the force of attraction, m_1 and m_2 are the masses separated by a distance r and G is the constant of proportionality. G is referred to as the (universal) constant of gravitation.

Law of Universal Gravitation

Every object in the Universe attracts every other object with a force directed along the line of centers for the two objects that is proportional to the product of their masses and inversely proportional to the square of the separation between the two objects.

$$F_g = G\frac{m_1 m_2}{r^2}$$



F_g is the gravitational force

m_1 & m_2 are the masses of the two objects

r is the separation between the objects

G is the universal gravitational constant

The importance of Calculus

- Before he could announce his Law, Newton had to show by the use of calculus (which he invented) that a symmetrical body (the Earth) could act as if all its mass were concentrated at its centre.

What is the Value of G?

- Newton had calculated the gravitational force that must act on the Moon for it to remain in orbit.
- If he knew the mass of the Earth he could calculate the value of G.
- The density of the Earth at the surface is 2700 kg/m^3
- He suspected that it got greater with depth so he doubled it to get the Earth's average density
- His estimate was 5400 kg/m^3
- Close to the actual value of 5520 kg/m^3

- The acceleration due to gravity at the distance of the Moon g_m is 0.00272 m/s^2
- From his second Law (force = mass x acceleration) the force acting on the Moon must have been:

$$F = M_m \times 0.00272 \text{ kg m/sec}^2$$

$$= M_m \times 0.00272 \text{ N}$$

- Equating this with the force as calculated form his law of gravity gives:

$G \; M_e \, M_m \, / \, d^2 \; = M_m \times 0.00272$

Giving $G = 0.00272 \times d^2 / M_e$

with d = 384,000 km and

$M_e = 5400 \times 4/3 \times \pi \times (6.4 \times 10^6)$kg

$= 5.93 \times 10^{24}$ kg

$G = 0.00272 \times (3.84 \times 10^8)^2 / 5.92 \times 10^{24}$ N m^2 kg^{-2}

$= 4.0 \times 10^{14} / 5.92 \times 10^{24}$ N m^2 kg^{-2}

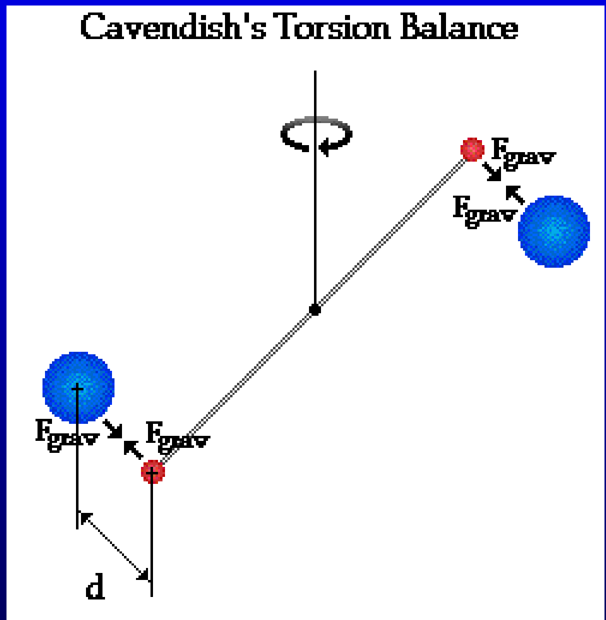
$= 6.76 \times 10^{-11}$ N m^2 kg^{-2}

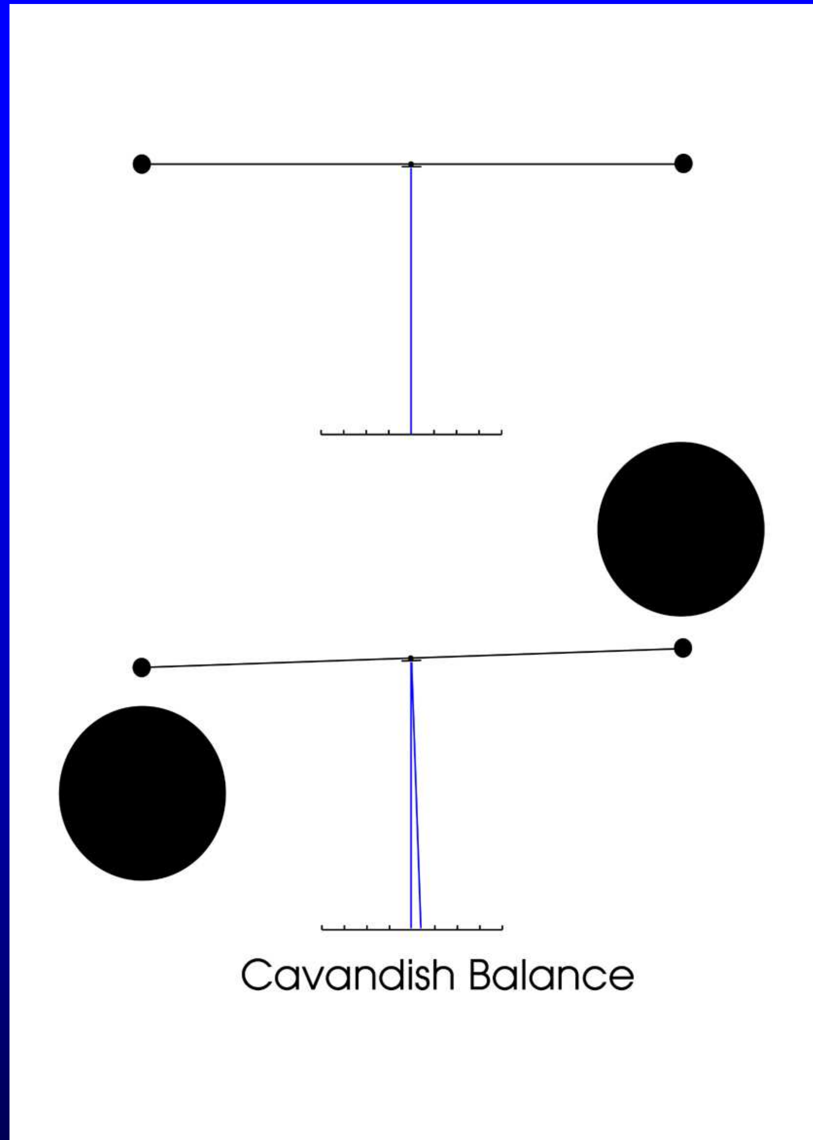
Due to his lucky estimate of the mean density of the Earth this was quite good.

The currently accepted value of G is

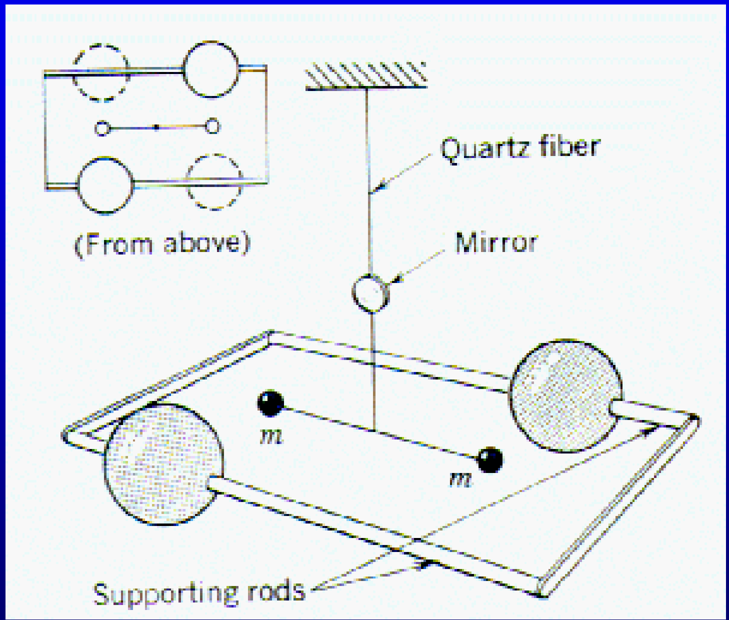
6.67×10^{-11} N m^2 kg^{-2}

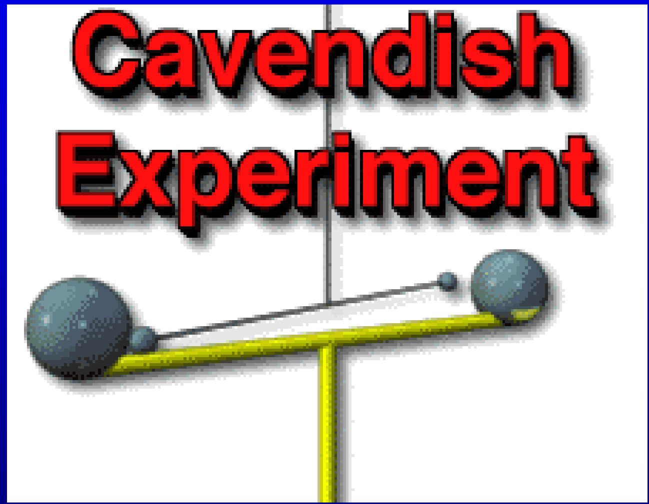
Experimental Method





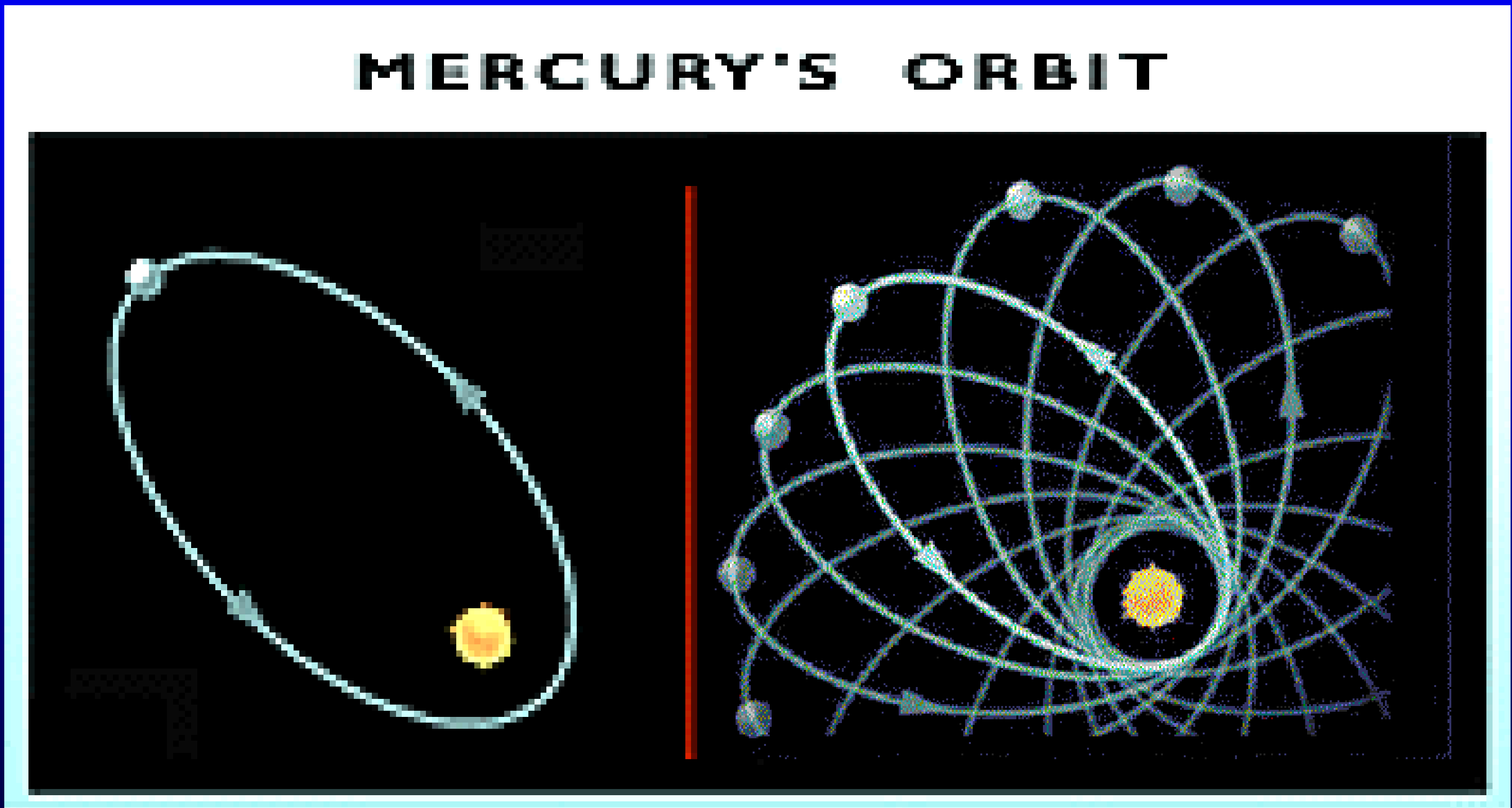
- G = 6.67×10^{-11} Nm^2kg^{-2}.


(From above)
Quartz fiber
Mirror
m
m
Supporting rods


Cavendish
Experiment

Current Apparatus

A problem - Mercury

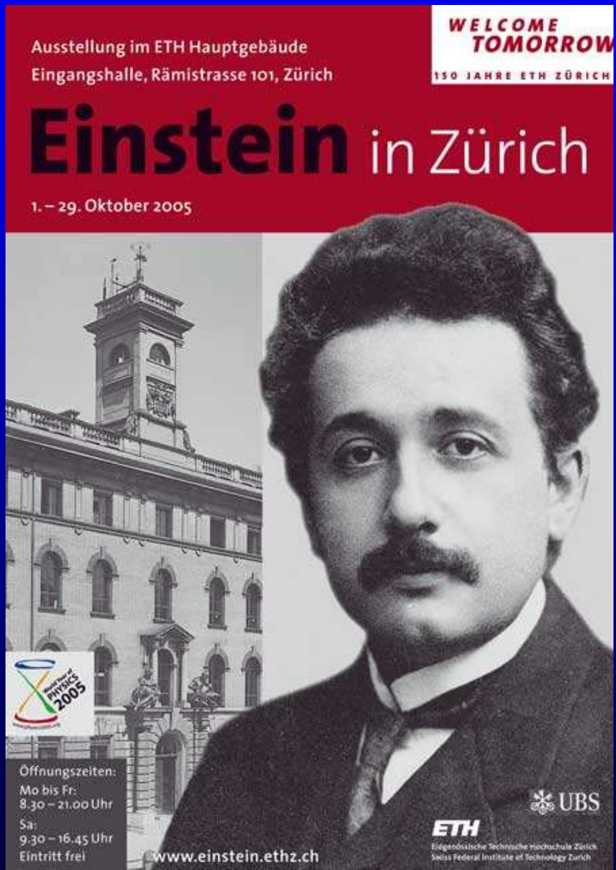


Precession of the major axis (or Perihelion) of the orbit

- The major axis of the orbit is seen to precess (rather like a spirograph drawing)

- The observed rate of precession was twice that derived from Newton’s Laws.

Albert Einstein



1905 – Annus Mirablis

- Three outstanding papers:
  - 1) Light is in the form of discrete particles – photons. The Photo-electric Effect. Awarded Nobel Prize for this work in 1921.
  - 2) The cause of Brownian Motion – a demonstration of the random motion of atoms.
  - 3) The Theory of Special Relativity – with a supplementary paper relating energy to mass.

The Special Theory of Relativity

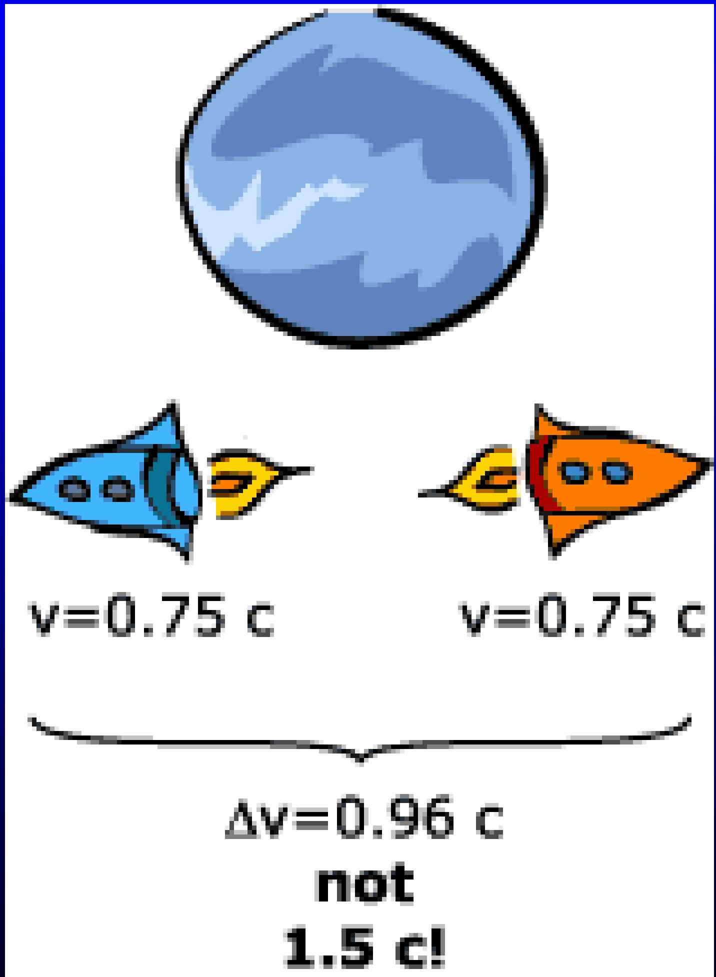
- Light travels through space at a constant speed c = 186,282 miles per second

  = 299,792,458 metres per second.

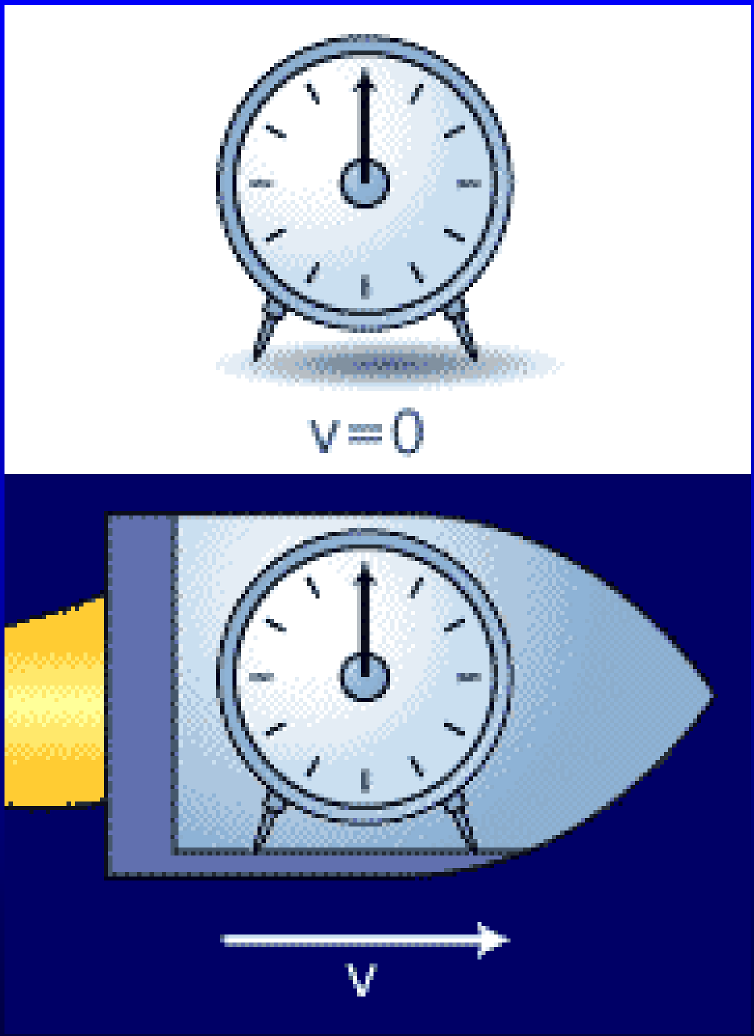
  It takes 3 nanoseconds to travel one metre

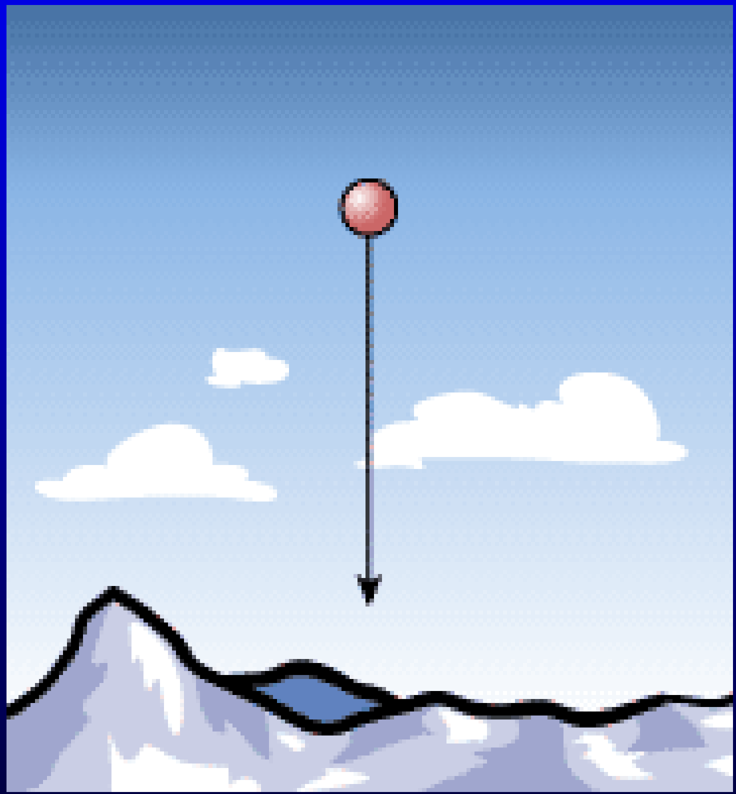
  No material thing or any information can travel through space faster than the speed of light.

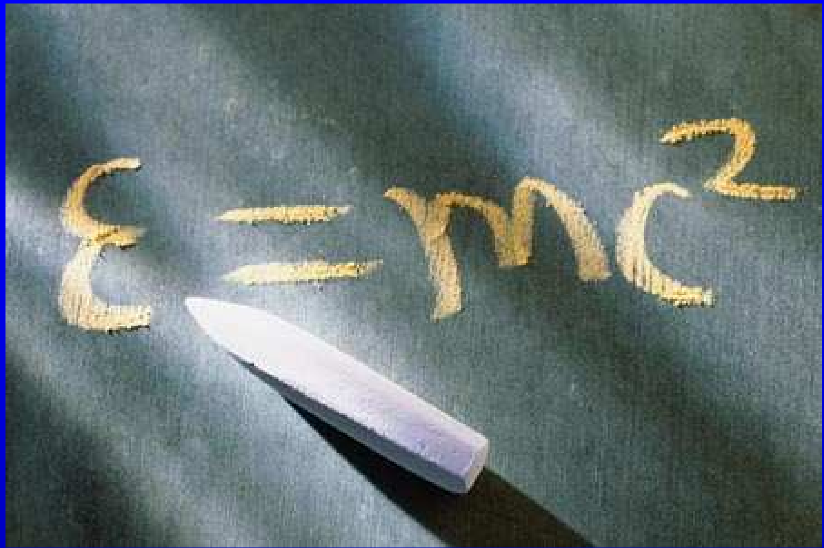
A more complicated formula for the summation of speeds

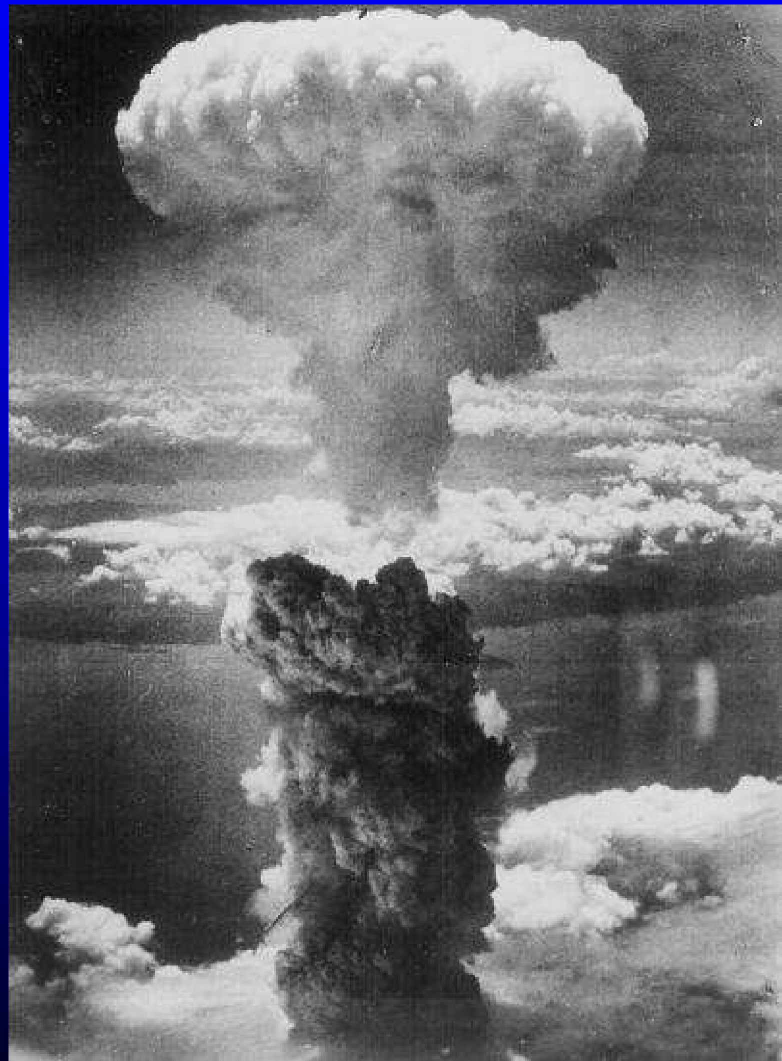


Time Dilation






E = mc2



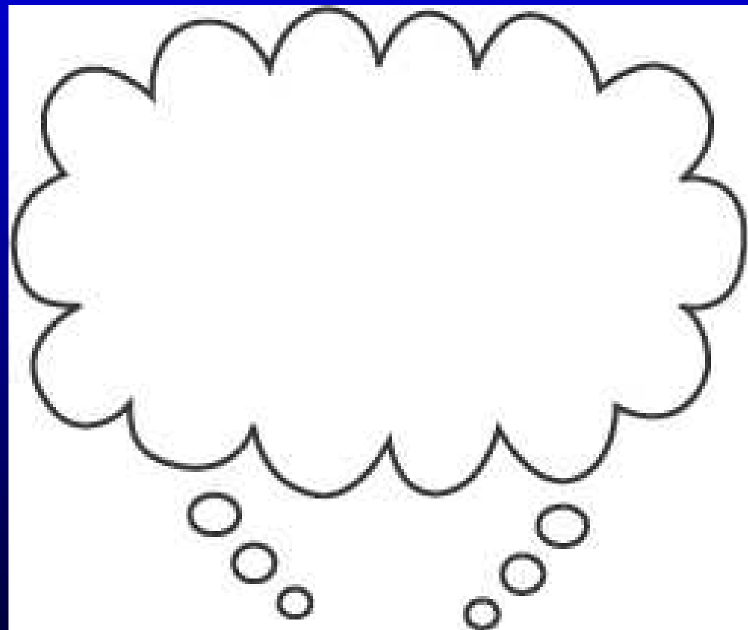
1915: The General Theory of Relativity *

- Really a theory of gravity.
- He realised that there was a fundamental problem with Newton's Theory.

* (This is often, wrongly, called the Theory of General Relativity. However it may simply be called "General Relativity")

A Thought Experiment

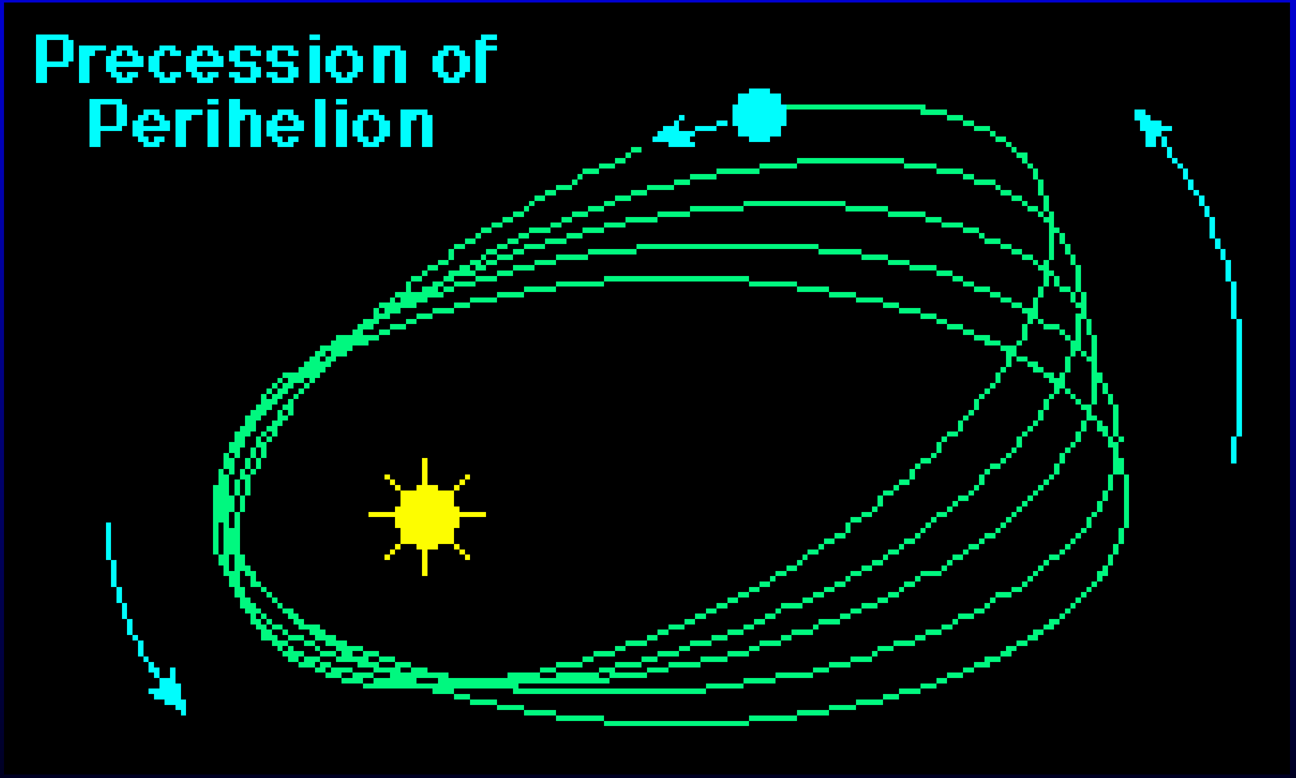
- Let's consider what would happen in the Sun suddenly disappeared.
- (Read the details in the book!)



Einstein’s Theory

- The effects of a change in mass distribution cannot propagate faster that the speed of light.
- These changes are carried by gravitational waves that travels through space at the speed of light.
- So he predicted the existence of gravitational waves.

Explains the error in the observed precession of the major axis of the orbit of Mercury

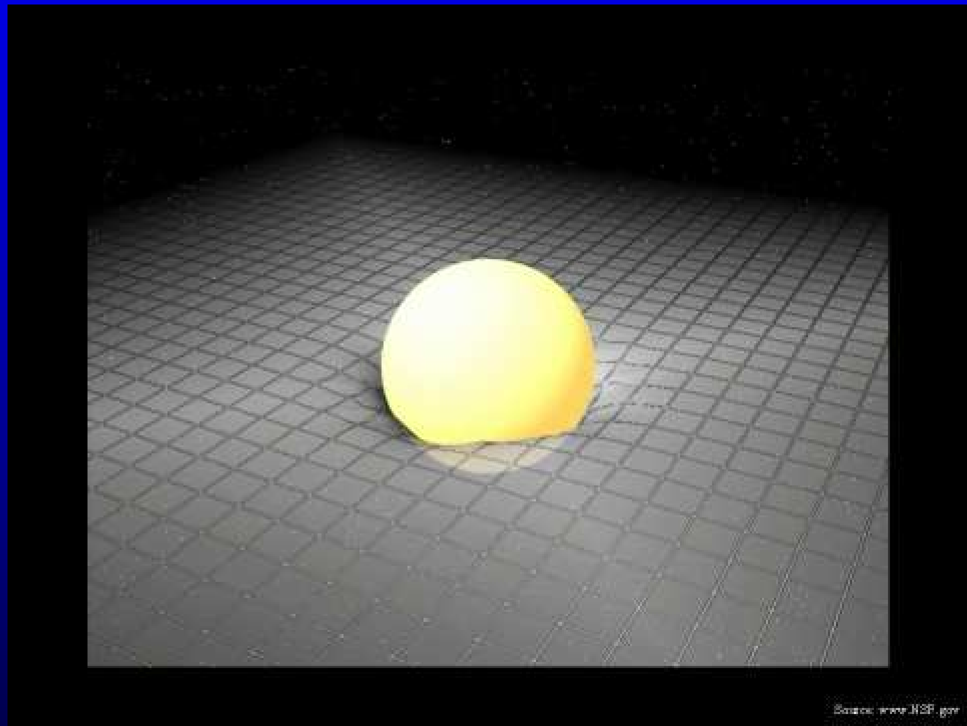


“Flat” Space

- In Einstein’s view, in the absence of mass space is “flat”.
- This means that Euclidian geometry holds true:
  - The inscribed angles in a triangle add up to 180 degrees
  - Two parallel laser beams a few metres apart will remain parallel

What is gravity?

- Einstein's Theory predicts that a mass will distort the "space-time" around it – giving the surrounding space positive curvature.



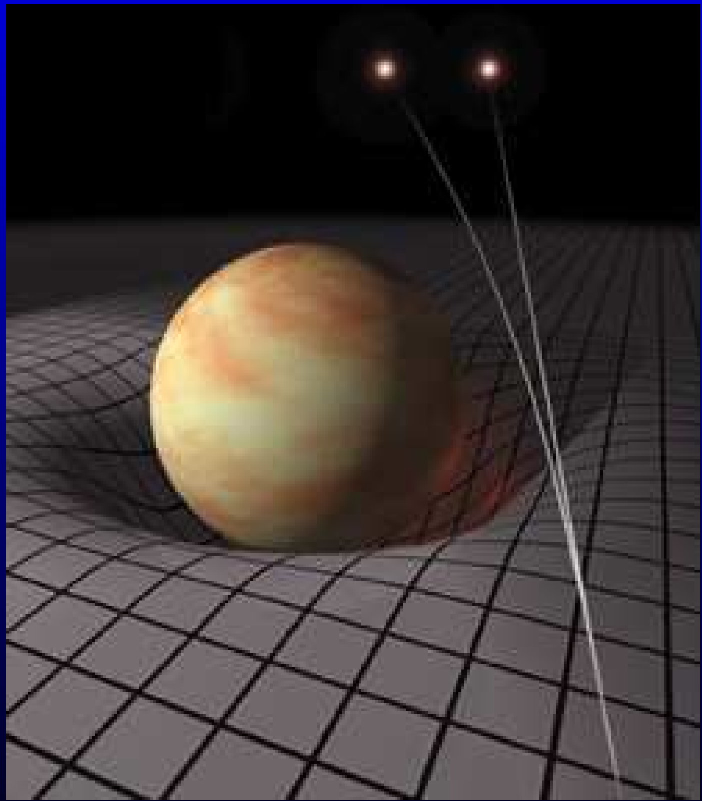
Positively Curved Space

- In positively curved space the inscribed angles of a triangle add up to more than 180 degrees.
- Two initially parallel laser beams will converge.

An Analogy

- Imagine a stretched rubber sheet (flat space)
- Ball bearings (photons) will roll across it in straight lines.
- If we put a lead ball on the sheet it will form a depression in the sheet – so curving the “space” around it.
- A ball bearing passing close by will follow a curved path – as would a photon in curved space.

In this region light will travel along curved lines



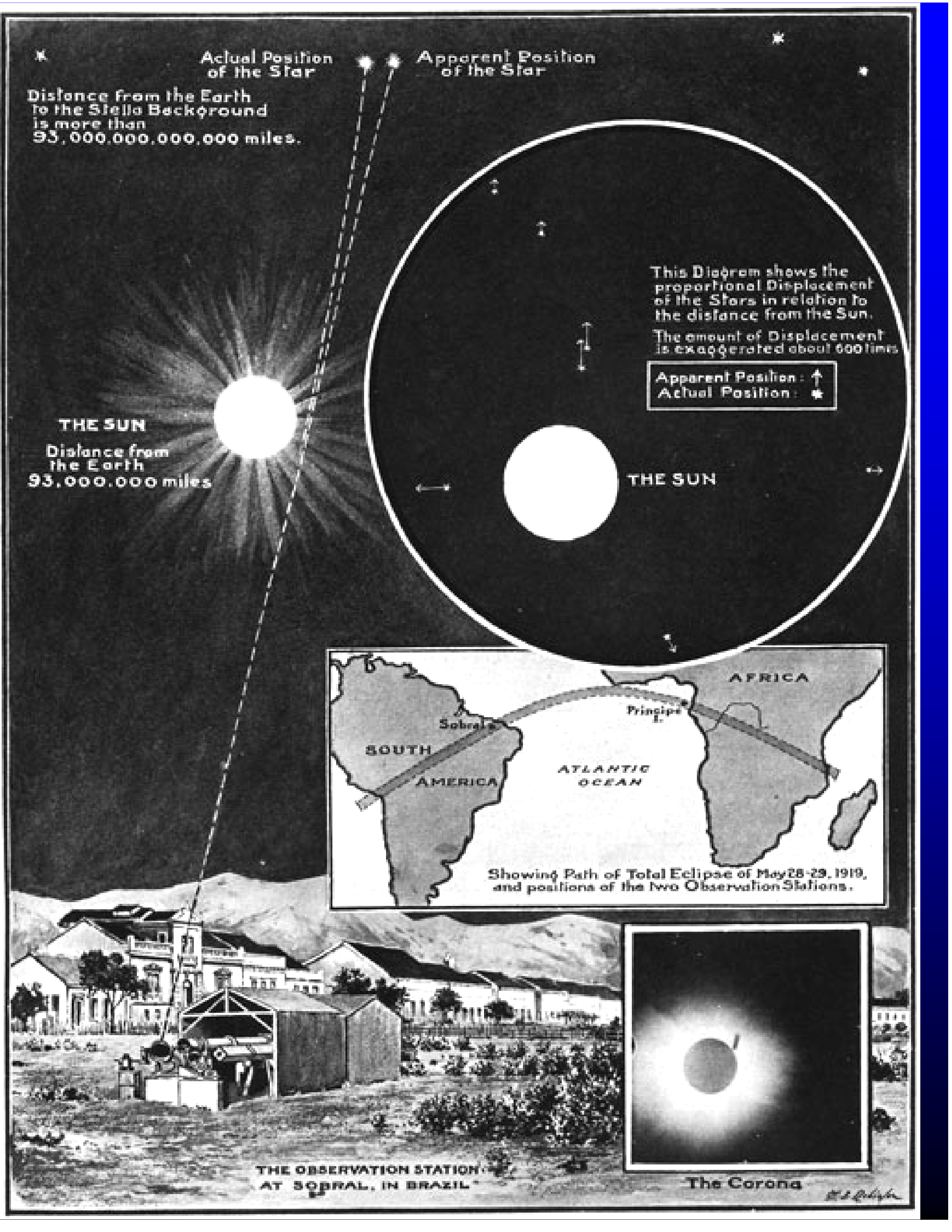
- In Einstein’s view the motion of the Earth around the Sun is simply the natural motion of a body through curved space.
- Gravity is a force that we infer to explain what happens in a curved three dimensional space if we assume that space is flat, not curved.

Eclipse Expeditions

to prove Einstein's Theory.

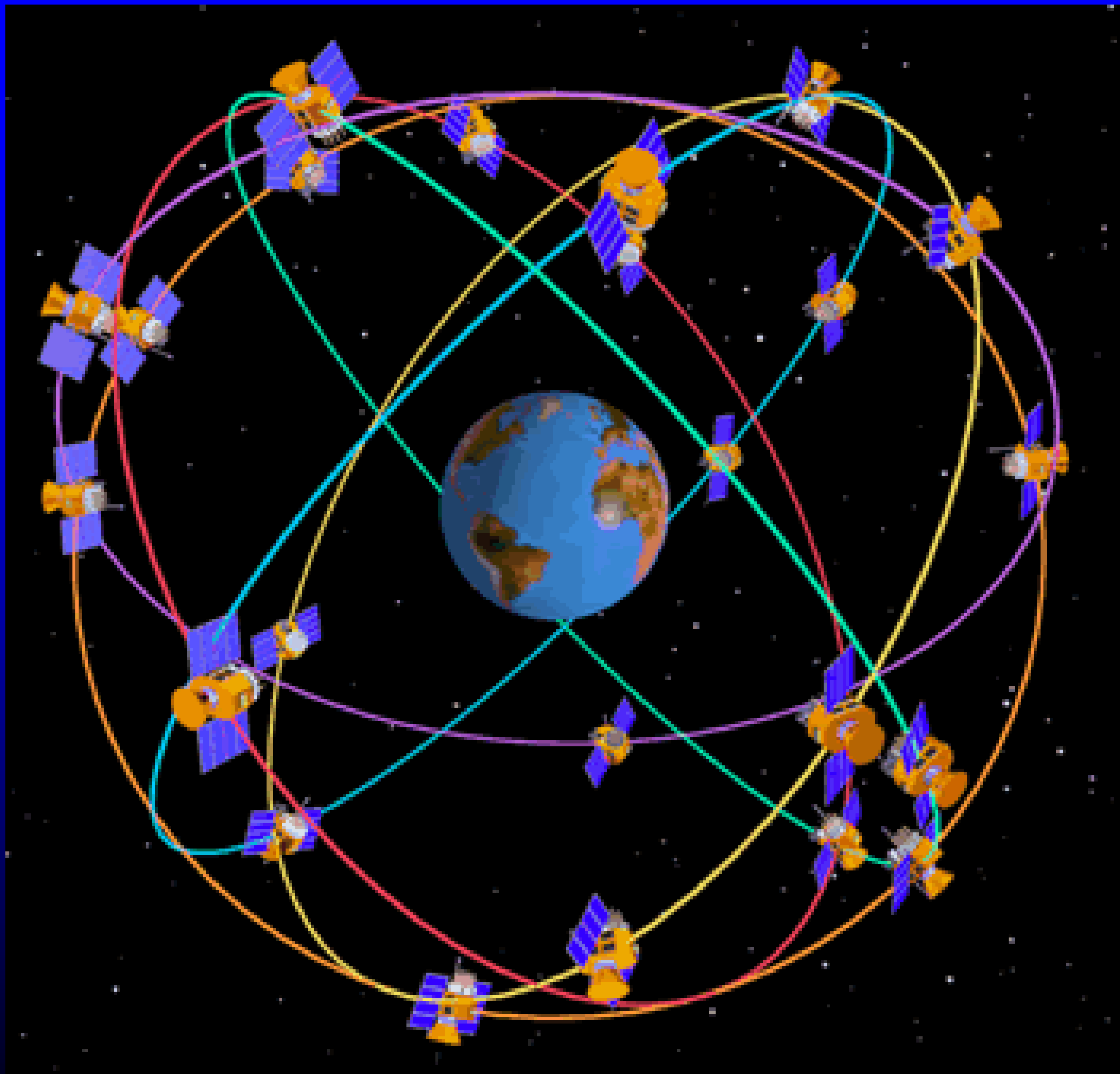
1919 dubious

1922 fine



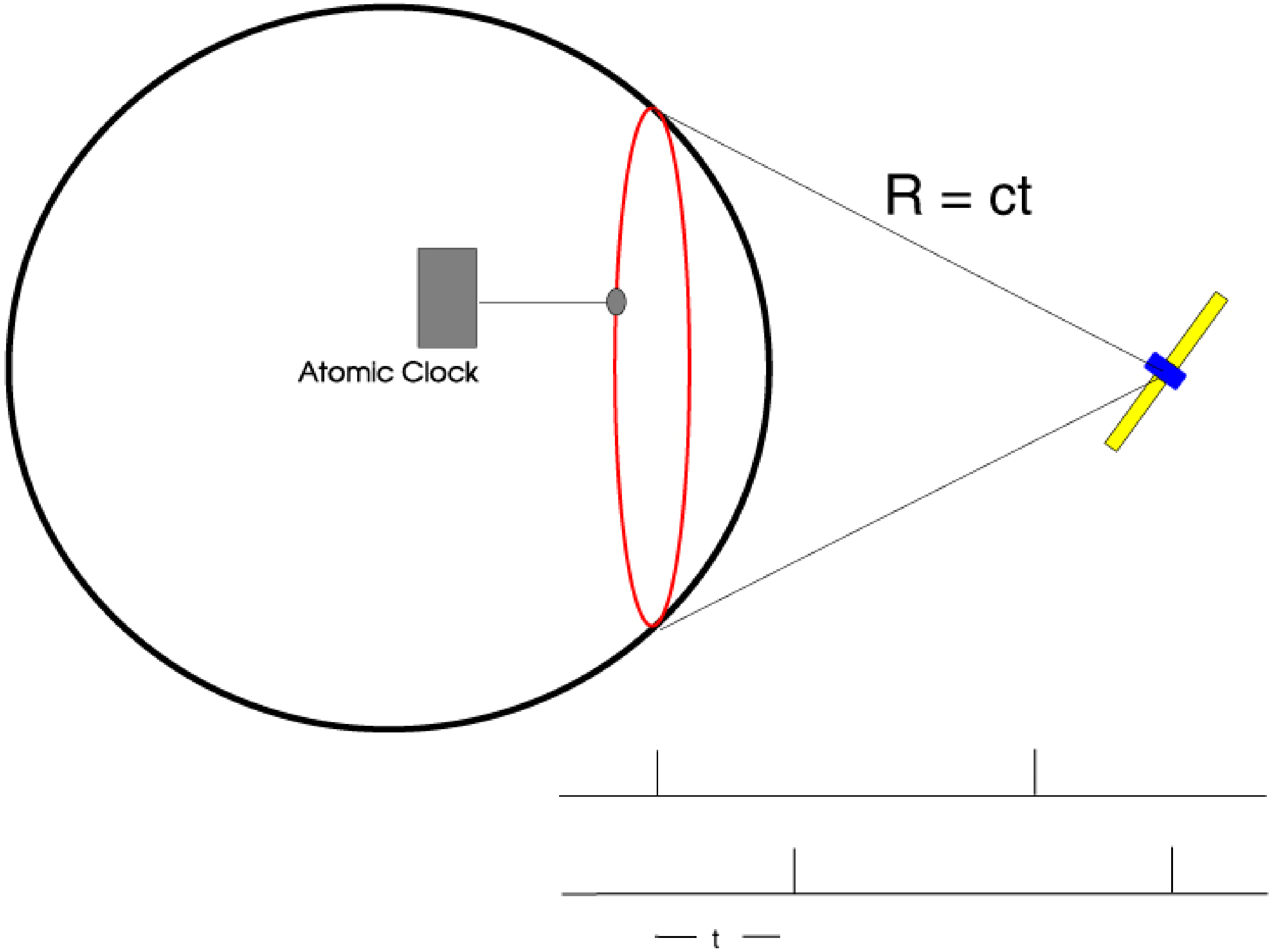
Einstein’s Theories in action

The Global Positioning
Network

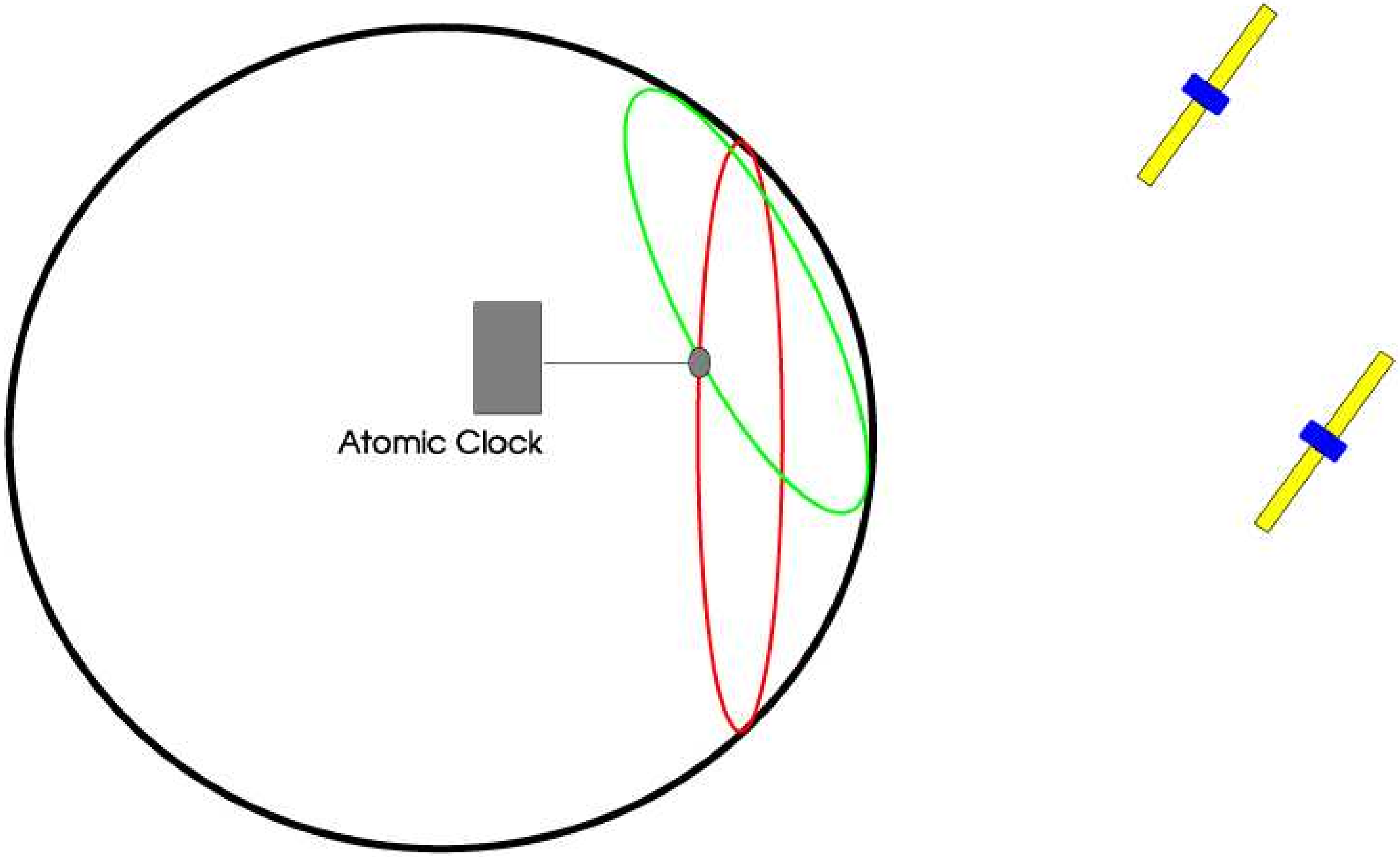


How it “works”

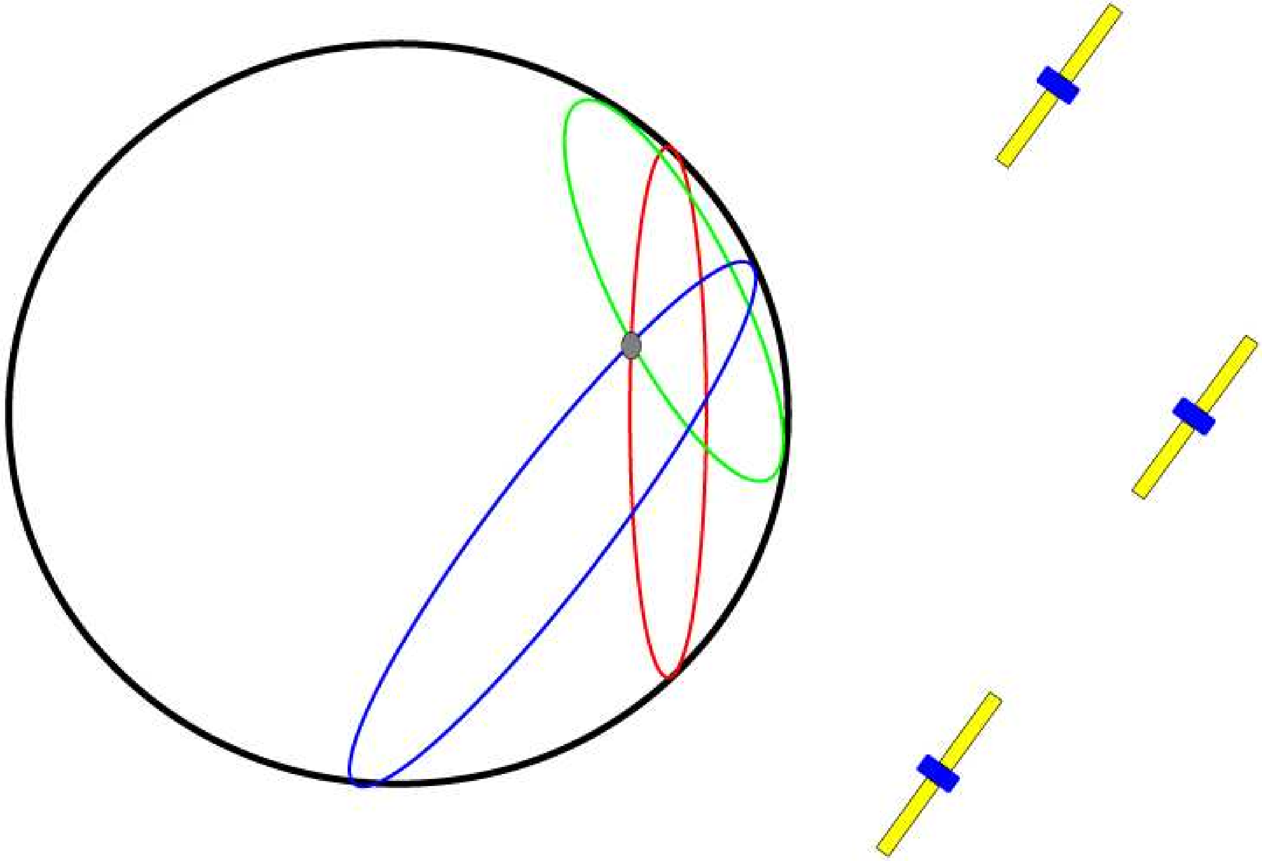
- With a single satellite and a receiver with a synchronised atomic clock one can measure the time a signal (assume a pulse each second for simplicity) takes to reach you. You must lie at a distance “ct” from the satellite which determines that you must lie on a circle round the globe – shown in red.


R = ct
Atomic Clock
t

- A second satellite defines a second circle.
- You must be at one of the two points where they cross.


Atomic Clock

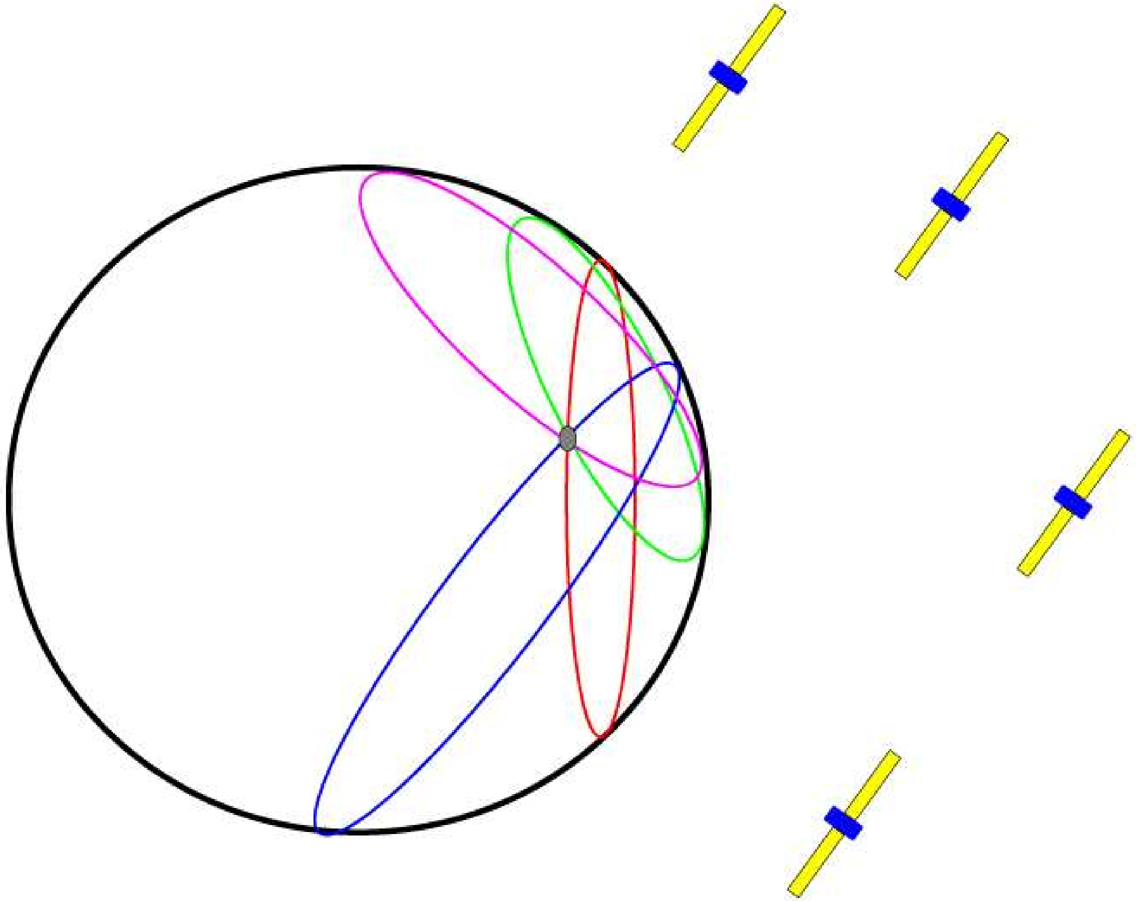
- With three satellites you do not need an atomic clock - the time can be solved.
- (This makes it much cheaper.)
- But you will only know position, not height.



Three Satellites

Give position only - but can solve for T

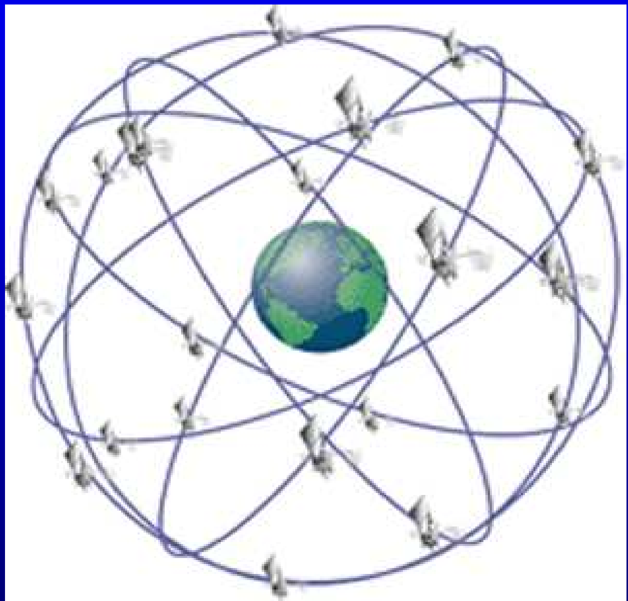
- Finally with 4 satellites one can get position and height.
- (Though height is not too accurate)
- More satellites improve accuracy.
- Now typically 8 will be in view from a clear site.



Four Satellites

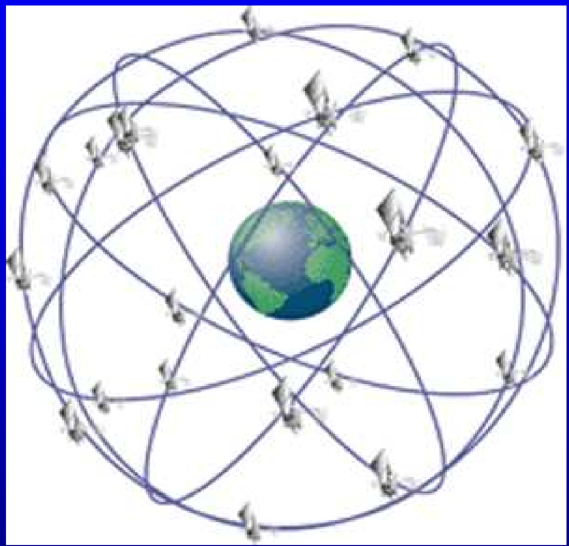
Give position and height - 3D navigation

Atomic Clocks in GPS satellites



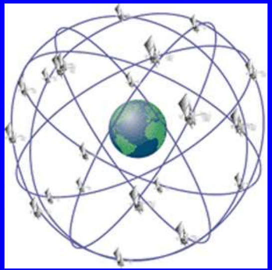
- As the Satellites are moving with respect to those on Earth they will appear to lose 7 microseconds a day **BUT …..**

Atomic Clocks in GPS satellites



- Einstein's General Theory of Relativity shows that clocks run more slowly in a stronger gravitational field. As the atomic clocks on the satellite are in a gravitational field that is only ¼ of that on Earth they will run faster by 45 microseconds per day.

GPS



- Taking account of the 7 microseconds slowdown relative to clocks on Earth due to the Special Theory of Relativity the GPS clocks would gain 38 microseconds a day.

- This would give an error of 10 km after one day!