1.3 Dynamics of the Universe

1.3.1 The Friedman Equation

We have seen that in a homogeneous, isotropic universe the only change allowed is an overall expansion or contraction, which is characterised by the scale length $R(t)$. The rate of change of R (the "speed" of expansion of the universe) is written $\dot{R}$, but it is usually more convenient to work with the Hubble parameter $H = \dot{R}/R$. If we could calculate *how* R changes with time we would know the whole past and future history of the Universe, at least on the largest scales. On these scales the only force that acts is gravity, and the best available theory of gravity, GR, gives the answer in the form of a surprisingly simple equation, named in honour of the Russian cosmologist Alexandr Friedman (1888-1925). The equation is:

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}$$

Here G is Newton's gravitational constant, ρ is the density of the universe (energy density $/c^2$), and k is the curvature constant, -1, 0, or +1 (see Section 1.2.4). Although Friedman himself derived the equation from GR, with a bit of work we can more or less see where it comes from by using only Newton's inverse-square law of gravity (see the following box if you are interested).

To solve the Friedman equation and predict the history of the Universe, we need to know:

- the present energy density in the Universe, $u(t_0) = \rho(t_0)c^2$
- the Hubble constant H_0, which, with $\rho(t_0)$, allows us to calculate the curvature constant k and $R_0 = R(t_0)$ which gives the scale on which geometry departs from Euclid,
- the way the energy density changes as the Universe expands, which depends on the type of "stuff" involved.

1.3.1.1 Derivation of the Friedman Equation

Recall that the gravitational force between two objects with masses M and m, separated by a distance r, is $F = -GMm/r^2$. This implies that the potential energy is $V = -GMm/r$. It is negative because we have to do work to pull the particles further apart (larger r).

As ever, we assume that the universe is perfectly homogeneous and isotropic. The idea is now to focus on the behaviour of a small region, which of course will behave exactly like every other region.

First, we surround our chosen region with a spherical surface of co-moving (dimensionless) radius χ. At any time t the physical radius will be $r = R(t)\chi$. We are going to ignore the rest of the universe beyond that distance because, being completely uniform in density, its gravitational pull acts equally in all directions and so cancels out. There is a famous proof by Newton which shows this in detail for the inside of a hollow sphere, and you can divide up the outer universe into a series of concentric spheres to see how to extend the idea to an infinite universe. Actually it is a bit of a cheat to extrapolate to infinity in Newtonian physics, but luckily in GR it can be rigorously proved that this is

OK.

Now consider two galaxies, one in the middle of our sphere and one on the outer edge. From Hubble's law, we know that at a given time they are moving apart at a speed $v = Hr$. Relative to the central one, the second galaxy has a kinetic energy $T = (1/2)mv^2$ where m is its mass, and a potential energy $V = -GMm/r$ due to the gravitational field of the matter within the sphere (mass M).

From conservation of energy, $T + V$ is constant with time. We can write the mass in the sphere M in terms of the average density of the universe, ρ, times the volume of the sphere. This gives

$$T+V=\frac{1}{2}mv^2-\frac{GM\rho}{r}\times\frac{4}{3}\rho\pi r^3$$

Both kinetic and potential energy are proportional to the galaxy mass m, and, on close inspection, to the square of the co-moving radius, χ; so we will say $T + V = Cm\chi^2$ where C is constant but so far unknown.

1. Show that T and V are both proportional to χ^2.
For answers to questions see the end of this document.

Dividing through by $mr^2/2$, and rearranging, we get

$$\left(\frac{v}{r}\right)^2=H^2=\frac{8\pi G}{3}\rho+\frac{2C}{R^2}$$

which is (almost) the Friedman Equation. The one problem is that the constant C is unknown. From Newtonian gravity this is as far as we can go. Einstein's theory supplies the constant:

$$2C = -kc^2.$$

Our Newtonian derivation assumed that the universe was composed of ordinary matter, so that mass was conserved when our sphere expanded. The real Universe did not behave like that at early times, and recent discoveries suggest that it does not behave like that at the moment. But it happens that the Friedman equation remains correct, provided we interpret the density using Einstein's mass-energy equivalence, $E = mc^2$. That is, the effective mass density is defined by $\rho = u/c^2$ where u is the density of *energy* in the Universe.

1.3.2 What happens when the Universe expands

We need to know how the density depends on the expansion of the Universe, which we can specify by the scale length. That is, we consider density to be a function of R: $\rho(R)$. Since the volume of any bit of the Universe is proportional to R^3, this is equivalent to finding a formula for density as a function of volume; such a formula is called an **equation of state**.

Exact equations of state tend to be very complicated, but luckily almost all the time the equation of state of the Universe is very close to a power law:

$$\rho(R) = u(R)/c^2 \propto R^{-n}$$

where the power n depends on what type of matter is involved. For this type of equation of state the pressure turns out to be

$$P = \left(\frac{n}{3} - 1\right)u = wu \quad .$$

Usually we specify the equation of state by the value of w rather than n; notice that $n = 3(w + 1)$.

1.3.2.1 Cold Matter

We know that a particle with energy E has mass $m = E/c^2$. This includes a component called the **rest mass**, m_0, which is always present, plus extra mass contributed by the kinetic energy of the particle. If the speed v is much less than the speed of light, the kinetic energy is $(1/2)m_0v^2$, which is much less than m_0c^2, so the energy is dominated by the rest mass. Since the speed of particles in a gas is controlled by the temperature, we say that the matter is **cold** if v « c. Even gas at 108 K is "cold" in this cosmological sense!

This is the case we assumed when deriving the Friedman equation. The energy in a region bounded by fixed co-moving co-ordinates (a **co-moving volume**) is the sum of the particle rest masses inside, which is constant; and so the density is just proportional to the inverse volume, i.e.

$$\rho_{matter} \propto R^{-3} \quad .$$

With $n = 3$, the pressure of cold matter is zero, or rather, negligible compared to the rest mass energy density.

The simple way to find the proportionality constant is to use the present-day values:

$$\rho_{matter}(R) = \rho_{matter}(R_0)\left(\frac{R_0}{R}\right)^3 = \rho_{matter}(R_0)(1+z)^3 \quad .$$

1.3.2.2 Radiation

Nearly all the photons in the Universe belong to the Cosmic Microwave Background. They were created soon after the Big Bang and, except for a negligible fraction which run into stars, radio telescopes, etc, they are not destroyed. Although individual photons move through the Universe, the number in a given co-moving volume stays the same because equal numbers enter and leave (always assuming homogeneity). Thus the number density of photons is proportional to R^{-3}, just as for the number density ordinary matter particles. But whereas the energy (= rest mass) of each matter particle is constant with time, we have seen that all photons suffer a redshift, so the energy of each photon, hc/λ, is proportional to $1/R$. Thus the radiation energy density is proportional to (density of photons) × (energy of each photon)

$$\rho_{radiation} \propto R^{-4} \, .$$

Radiation has a pressure $P = u/3$. These equations are exactly true for photons, but are also approximately correct for any particles which are moving close to the speed of light, in which case special relativity theory tells us that their energy is much greater than their rest mass energy. For particles at temperature T, the typical energy per particle is $\sim k_B T$, where k_B is Boltzmann's constant. Thus any kind of particle which is "relativistically hot", i.e. $k_B T \gg m_0 c^2$, counts as "radiation" as far as the Friedman equation is concerned.

1.3.2.3 Dark Energy

Dark energy is a generic term for any kind of stuff with *negative* pressure.

The prototype for dark energy was the **cosmological constant**, usually represented by the Greek letter Λ (Lambda), corresponds to an effective density that is independent of R:

$$\rho_A = \frac{\Lambda c^2}{8\pi G} = \text{constant i.e.} \propto R_0 \quad .$$

This implies $w = -1$, and hence, for positive Λ, negative pressure: $P = -u$. In principle, Λ might be negative, which would not strictly count as dark energy; but observations suggest a positive value, if any.

To see how peculiar this idea is, think of a cylinder containing "Λ stuff" (Fig. 1.11). If you pull out the plunger, increasing the volume filled by the Λ stuff, you will have increased the energy in the cylinder, as the density is unchanged and the volume is bigger. By conservation of energy, you must have done some work; in other words, the Λ stuff must pull back on the plunger; it corresponds to a sort of *tension* in space. This is the meaning of negative pressure.

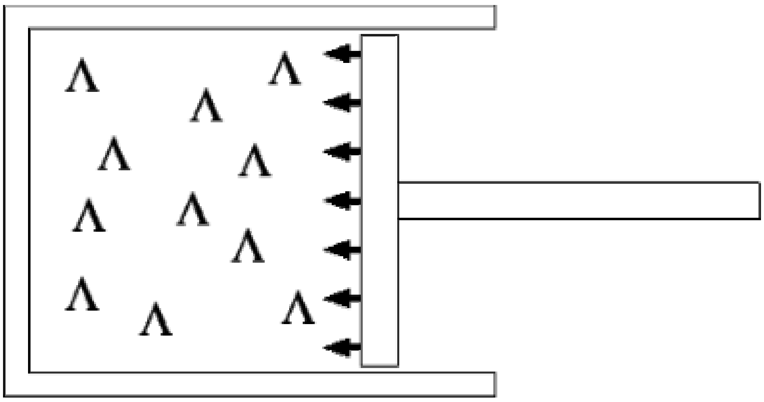

Figure 1.11: *A cylinder full of 'Cosmological constant' will exert a negative pressure on the plunger.*

The original motivation for this strange idea was that in 1917 Einstein found that, according to GR, a universe containing only ordinary matter would inevitably expand or contract. In an unusual failure of imagination, he refused to consider the possibility that this might be happening, and instead proposed the cosmological constant to keep the universe static.

For obvious reasons the cosmological constant fell into disfavour when the expansion of the Universe was discovered; but since the early 1990s, observations have increasingly seemed to be inconsistent with a universe dominated by matter, and a cosmological constant seems to give the best fit to the data, even though it is still as *ad hoc* as ever.

The idea of a cosmological constant irritates many theorists, partly because a universal energy density which is strictly constant in time and space has almost no effect on anything except the large-scale expansion of the universe. This means that it is very difficult to imagine ways of getting corroborating evidence for its existence. It also means that there is no scope for elaborating the theory.

Therefore there is a lot of interest in the possibility of a component of the Universe which behaves something like a cosmological constant, but has more "interesting" properties. This something has been named **quintessence**.

2. Look up "quintessence" in a dictionary. Is it a well-chosen name?

Quintessence is a material with equation of state

$$P_Q = w\rho_Q$$

with -1 < w < 0. This implies that the density depends on R according to:

$$\rho_Q \propto R^{-3(w+1)}.$$

Our earlier discussion shows that w = -1 corresponds to a cosmological constant, and w = 0 corresponds to ordinary matter. Unlike the cosmological constant, both the density and (in some versions) w may vary from place to place on small scales, just as ordinary matter does. While all this makes life interesting again for theorists, observations cannot yet distinguish quintessence from the vanilla-flavour cosmological constant, and so we will mostly ignore this possibility.

1.3.3 Model Universes

1.3.3.1 Density Parameters (Ω)

According to the Friedman equation, we can deduce the geometry of the universe if we know the Hubble parameter and the density. If the universe has the **critical density** ρ_c such that $H^2 = 8\pi G\rho_c/3$, we must have k = 0. If the density is less than critical, $-kc^2/R^2$ must make up the difference, so the universe must be negatively curved, k = -1; correspondingly, if the density is higher than ρ_c the curvature must be positive, k = 1.

3. Calculate the present value of ρ_c assuming that H_0 = 100 km s^{-1} Mpc^{-1}. Work out your answer in (a) kg m^{-3}, (b) protons m^{-3}, (c) Galaxies per cubic Mpc (take 1 Galaxy = 10^{12} solar masses).

In cosmology, we quantify density in terms of its ratio to the critical density; this ratio is known as the *density parameter*:

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)}.$$

Note that ρ_c varies with time because H does.

In some cases the major uncertainty in Ω is the uncertainty in ρ_c caused by our inaccurate knowledge of H_0. For that reason, we often use a fudge factor $h = H_0/(100\ \mathrm{kms^{-1}Mpc^{-1}})$. Then we can write $H_0 = 100h\mathrm{kms^{-1}Mpc^{-1}}$, and let h propagate through the equations so that at any time we can insert our favourite value, e.g. $h = 0.72\pm0.08$ was recently derived using the Hubble Telescope. As h is dimensionless, we also avoid having to write out the units for H_0! For instance, given a physical density ρ, we can unambiguously find a value for $\Omega h^2 = \rho/(\rho_c/h^2)$. (It should be obvious from the context when h means the scaled Hubble constant and when it means Planck's constant).

As usual, the present value of $\Omega(t)$ is written $\Omega_0 = \Omega(t_0)$. Just as the total density can be found by totting up the densities in the individual components, we can define density parameters for matter, radiation, etc, which sum to give the total density parameter:

$$\Omega_0 = \frac{\rho(t_0)}{\rho_c(t_0)} = \frac{\rho_{matter}(t_0)}{\rho_c(t_0)} + \frac{\rho_{radiation}(t_0)}{\rho_c(t_0)} + \frac{\rho_\Lambda}{\rho_c(t_0)} + \ldots = \Omega_m + \Omega_r + \Omega_\Lambda + \ldots$$

By convention the density parameters for individual components on the right hand side of the above equation usually refer to the present time, so we don't have to use double subscripts. We can calculate the values at other times using the equations of state from Section 1.3.2:

$$H^2\Omega(t) = \frac{8\pi G}{3}\rho(t) = \left(\frac{H_0^2}{\rho_c(t_0)}\right)\rho(t) = H_0^2\left[\Omega_m\left(\frac{R_0}{R}\right)^3 + \Omega_r\left(\frac{R_0}{R}\right)^4 + \Omega_\Lambda\right].$$

Inserting this into the Friedman equation gives the fairly horrible looking

$$H^2 = H_0^2\left[\Omega_m\left(\frac{R_0}{R}\right)^3 + \Omega_r\left(\frac{R_0}{R}\right)^4 + \Omega_\Lambda + (1-\Omega_0)\left(\frac{R_0}{R}\right)^2\right].$$

The $(1-\Omega_0)$ term results from substituting for the kc^2 term, due to the curvature of the universe; it is often written Ω_k.

4. Derive the above equation.

1.3.3.2 Simple Examples

Our equation is not as nasty as it looks because the various different factors of (R_0/R) mean that at any given time, one term in the bracket on the right is probably much larger than all the rest, which can then be ignored.

At very early times (high redshifts) $R_0/R \rightarrow \infty$. The radiation term grows fastest, so however small it is now, there was a time in the past when it dominates. In the early Universe, only radiation is important. Similarly, in the future, $R_0/R \rightarrow 0$ and both matter and radiation will become insignificant. The cosmological constant will dominate, if it exists; otherwise the curvature term takes over.

So for most of the time the Friedman equation looks like

$$H^2 = H_0^2\Omega_x(R_0/R)^n = H_0^2\Omega_x a^{-n},$$

where we have brought back the dimensionless scale parameter $a = R/R_0$. Furthermore, any time the curvature term is negligible we have $\Omega(t) \approx 1$, so if the above equation applies today we have $\Omega_x \approx \Omega(t_0) \approx 1$. If the curvature term dominates, $\Omega_x = \Omega_k = (1-\Omega_0)$, but in this case Ω_0 is very small and so again we can set $\Omega_k = 1$. Taking the square root of the equation, and remembering that $H = \dot{a}/a$, we have

$$\dot{a} = H_0 a^{1-(n/2)}$$

(we take the positive square root when the Universe is expanding). If you know some calculus, you will recognise this as a simple differential equation. The solution is

$$a(t) = \left(\frac{n}{2}H_0 t\right)^{2/n}$$

We can find the present age of the Universe t_0 in terms of the Hubble time $t_H = 1/H_0$, since then $a(t_0) = 1$:

$$1 = \frac{n}{2}\frac{t_0}{t_H} \quad \Rightarrow \quad t_0 = \frac{2}{n} t_H .$$

For instance, if ordinary matter dominates, we have $t_0 = (2/3)t_H$, and

$$a = \frac{R}{R_0} = \left(\frac{3}{2} H_0 t\right)^{2/3}$$

Since R is increasing more slowly than t, the expansion is decelerating; it is being opposed by the gravitational pull of the matter. This is called the **Einstein-de Sitter** model universe; it corresponds to $\Omega_m = \Omega_0 = 1$.

5. Find the equation for R/R_0 when radiation dominates.

If the curvature term dominates, $n = 2$, and we have a steady expansion, $R \propto t$. This makes sense as now all the matter terms are negligible and there is nothing to slow down or speed up the expansion. This is called the **Milne** model, for which $\Omega_0 = 0$.

In the same way, if the universe is dominated by a component with $n < 2$, which corresponds to $w < -1/3$ (c.f. Section 1.3.2), then the expansion will accelerate. This corresponds to some form of dark energy. This is a very counter-intuitive result. We saw earlier that the negative pressure of dark energy acts like a tension in space. You might expect this to pull the galaxies towards each other, slowing down the expansion like gravity; but in fact it makes them accelerate ever faster away. What is happening is first, that the actual tension (or negative pressure) has no direct effect because it is the same everywhere; each galaxy is being pulled equally in all directions. Second, in General Relativity the source of gravity is not just matter density but the combination $\rho + 3P/c^2$, where P is the pressure; we can also write this as $(1 + 3w)\rho$. This is implicit in the Friedman equation, and is the reason that for $w < -1/3$ we effectively have negative gravity, or "anti-gravity" if you prefer.

If the cosmological constant dominates we have a special case as $n = 0$. The Friedman equation shows that then the Hubble parameter is constant in time; the speed of a given galaxy increases in proportional to its distance from us, so the distance increases at an increasing rate. The result is exponential expansion:

$$\frac{R}{R_0} = \mathrm{e}^{H_0 (t - t_0)}$$

This is called the **de Sitter** model, with $\Omega_\Lambda = \Omega_0 = 1$.

1.3.3.3 Friedman-Lemaître Universes

Most of the time the universe can be described by the simple solutions of the previous section, but we need to look at more general cases to get the whole picture. These were first studied by Friedman, and later in more depth by Georges Lemaître.

Although the matter and radiation densities must obviously be positive numbers, this is not necessary for the cosmological constant and curvature terms, i.e. Ω_Λ and $(1 - \Omega_0)$ could be negative. As these terms become important for large R, if the larger of the two is negative it will eventually cancel the matter and radiation terms, so H^2 becomes zero. In other words, the expansion stops. After this time the universe slowly begins to recollapse. As far as the Friedman equation is concerned, the collapse is just the expansion run backwards (same equation, but take the negative square root to get H). After some speculation to the contrary, careful analysis of the equations of GR shows that this does not mean that time itself starts to run backwards; in a universe at turnaround clocks would continue to tick forward, people would continue to age, and entropy continues to increase; but the galaxies would start to show blue-shifts instead of redshifts.

In the present Universe we know that radiation is unimportant, so that to a good approximation $\Omega_0 = \Omega_m + \Omega_\Lambda$ (ignoring the possibility of quintessence, which in practice behaves much like Λ). Apart from an overall scale, set by H_0, the history of our Universe depends only on Ω_m and Ω_Λ. Applet 1.12 will calculate $R(t)/R_0$ for you, for any reasonable combination of these two parameters.

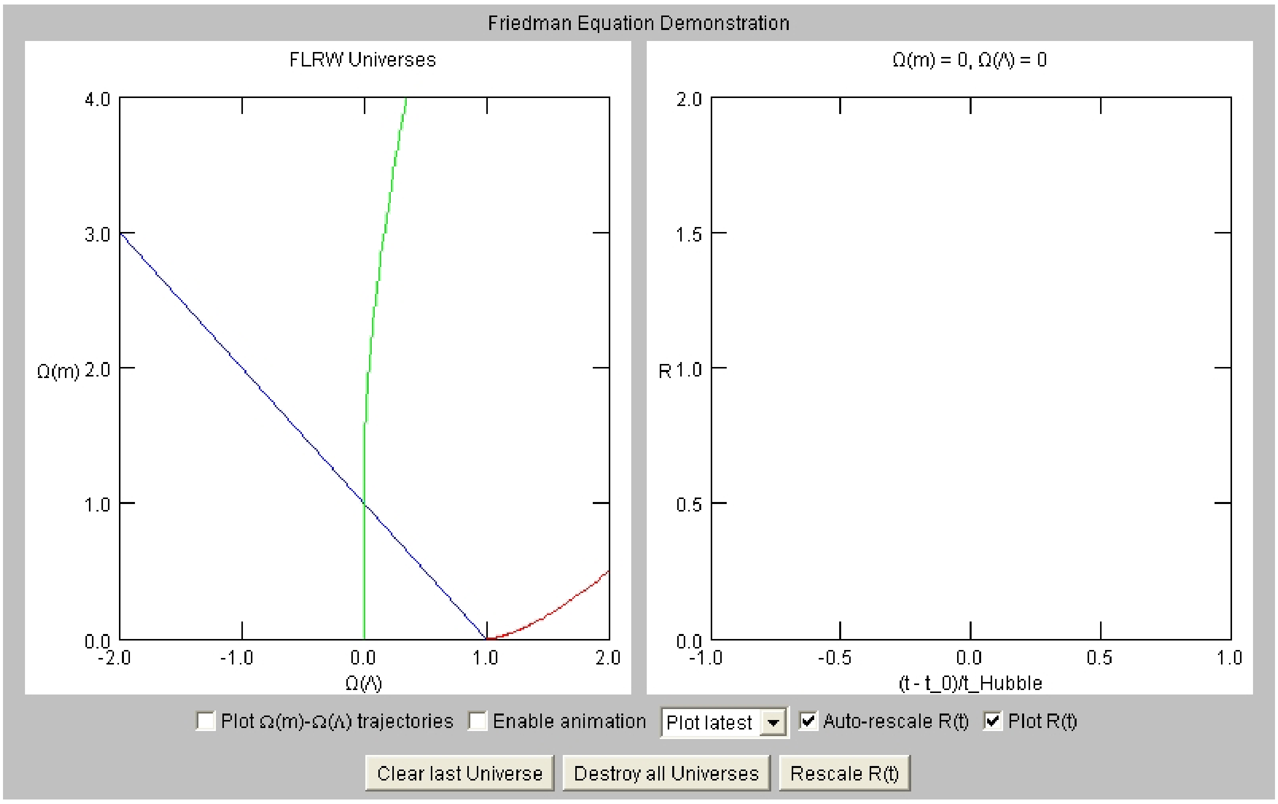

Figure 1.12:Screenshot from the Friedman applet – click here to open a web page containing the applet.

The Ω_m, Ω_Λ diagram on the left-hand side of Applet 1.12 is frequently used to show which combinations are currently allowed by the data. Getting to know this diagram and the various corresponding $R(t)$ curves is a good way to master the physics of the Friedman equation.

6. Current data suggest that $\Omega_\Lambda \approx 0.7$, $\Omega_m \approx 0.3$. Use the applet to plot $R(t)$ for this universe and hence read off the present age of the universe t_0 in terms of the Hubble time t_H.

Although the convention is that Ω_m and Ω_Λ normally refer to the present time (t_0), it is quite illuminating to see how they change, and Applet 1.12 will show you this. Clicking on the graph shows that nearly all universes start at the Einstein-de Sitter point with $\Omega_m = 1$ and $\Omega_\Lambda = 0$. This is because if there is a Big Bang (i.e. R goes to zero at $t = 0$) then ρ, hence H, hence ρc, all tend to infinity at time zero. So Ω_Λ tends to zero, while the matter density tends to the critical density.

The coloured lines divide the plane into six regions, in each of which the universe has a rather different character.

The blue line shows where $\Omega = 1$, i.e. $k = 0$. Above this line, space is positively curved (and closed), below it, it is negatively curved, and open in the simplest topology. Evolution tracks never cross the blue line because k cannot change.

The line $\Omega_\Lambda = 0$ separates universes with negative and positive cosmological constant. Negative values mean that the universe always recollapse eventually: the expansion halts and reverses at a certain point. Since at turnaround $H = 0$ by definition, the critical density at turnaround is zero, so the evolution tracks of universes with turnaround extend out to infinite Ω on both axes.

The green and red lines bound universes with positive cosmological constants that expand continuously from $R = 0$ to infinity. They start from the matter-dominated Einstein-de Sitter point and end at the Λ-dominated de Sitter point $\Omega_m = 0$, $\Omega_\Lambda = 1$.

The limiting case for an expanding, closed universe is the loitering universe, first studied by Lemaître. This corresponds to the curved green and red lines. Loitering universes reach a point with $H = 0$ and also zero acceleration, known as the **Einstein** solution (which lies at infinity on this graph because the critical density is zero). Along the green line the Einstein point is in the future, while it is in the past along the red line. At the Einstein point the repulsive negative pressure from the cosmological constant exactly balances the gravitational attraction of mass/energy:

$$\rho + 3P/c^2 = \rho_m + \rho_\Lambda - 3\rho_\Lambda = 0 \quad \Rightarrow \quad \rho_m = 2\rho_\Lambda .$$

This is a delicate and unstable balance. If the density is slightly too high, the universe will collapse from the Einstein point, following the green line down to a Big Crunch at the Einstein-de Sitter point. If the density is too low, the expansion takes off and the universe follows the red line to the de Sitter point of total domination by the cosmological constant.

In universes with positive cosmological constant, but which fall to the left of the green loitering line, the repulsive term is not enough to prevent recollapse, so they expand from the Einstein-de Sitter point to a turnaround at infinity, and then recollapse back to a big crunch along the same track. Thus all the tracks to the left of the green line expand from a big bang and recollapse to a big crunch.

Finally, universes to the right of the red loitering line have a cosmological constant so large that, projecting the universe back in time, the expansion turns around in the past; these are the bounce models, in which the universe collapses from infinity (contracting from the de Sitter point), bounces (both Ωs are infinite at the bounce as $H = 0$ here, just as at turnaround in "bang-crunch" universes), and then re-expands, finishing in an expanding de Sitter phase. Note that the limiting case here is when the universe collapses from infinity to bounce at exactly the condition for an Einstein universe, thereby following the red curve away from the de Sitter point.

1.3.3.4 Open vs. Closed Universes

Cosmologists often talk about open and closed universes. Unfortunately the meaning of these terms has become confused. The founders of modern cosmology, Einstein and Friedman, used them, as we do in this course, in their topological sense. They noted that the simplest topologies associated with positive and negative curvature were closed and open respectively. The possibility of more complex topologies was ignored by their immediate successors, and eventually almost forgotten. At the same time, the cosmological constant became unfashionable, and negative-pressure matter like quintessence was not considered. Without these terms the Friedman equation implies that a negatively curved universe expands forever and a positively curved one recollapses. Eventually cosmologists came to assume that "open" meant negatively curved and forever expanding, while "closed" meant positively curved and with a finite future – the slogan was that "geometry implies destiny".

This slogan is simply false; it is based on a naive interpretation of the Friedman equation which took far too much for granted. As we have seen, geometry (i.e. curvature) does not necessarily define the topology. If there is a cosmological constant, then universes with negative, zero, or positive curvature can all expand forever or recollapse; which actually happens depends on Ω_Λ and Ω_m. And, of course, this all assumes that the universe is genuinely homogeneous and isotropic on the largest scale.

7. What is the condition on Ω_Λ for a Euclidean universe to recollapse? (Make reference to the applet).

Some cosmologists, rather than admit they have been confused in the past, now prefer to *define* "open" and "closed" to mean "expands forever" and "will re-collapse". So treat these words with suspicion when you run into them in popular accounts, textbooks, or research articles.

1.3.4 Horizons

The largest distance that is visible in principle at a given time is called the **particle horizon**. There is such a limit because light travels at a finite speed; thus the particle horizon is the co-moving distance that light can travel since the beginning of the Universe. Real photons are obstructed by matter at very early times, so we had better make clear that for present purposes "light" is assumed to travel unimpeded.

The particle horizon seems an obvious concept, but mathematically it can be avoided if the universe expands so fast at the start that light can travel an infinite co-moving distance. This would only happen in a universe containing no matter or radiation to decelerate it, so it is not very realistic! But by the same token, the particle horizon is always further away that ct_0, because the universe expands as the light travels through it. For instance, for the Einstein-de Sitter case (Ω_m = 1, Ω_Λ = 0), the particle horizon is

$$R\chi_{ph} = 2ct_H = 3ct_0 .$$

The finite particle horizon is the solution to the paradox that the sky is dark at night (Olbers' paradox). See Strobel's notes for more about this.

At a given time, there may also be a largest distance that will ever be able to see *us*. This is called the **event horizon**. It is the co-moving distance that light can travel from now until the end of the Universe.

At first glance we might expect an event horizon only if the universe re-collapses. Otherwise, it lasts for ever and it is natural to expect that our light will eventually reach a galaxy at any distance at all. But in fact, again, the universe may expand too fast. If there is a positive cosmological constant (which seems to be true for our Universe), then we eventually reach the exponentially-accelerating de Sitter phase. Exponential acceleration means that any galaxy will eventually be receding from us faster than light, so light will never reach it. Photons emitted today will pass many galaxies, but as they travel and the universe expands, the gulfs between galaxies widen. Fewer and fewer galaxies are passed with each billion years. Eventually a last galaxy is reached. Observers there may see a redshifted galaxy yet further away, but this light carries old news. Since then, that galaxy has accelerated past the speed of light and our photon will never reach it.

In a universe like ours, with both particle and event horizons, there is a maximum co-moving distance that a photon can travel in the whole of time, equal to the sum of the two horizon lengths. Weirdly, if $\Omega > 1$ and $\Omega_\Lambda > 0$, the universe is spatially closed (and so finite in size), lasts forever, and yet may contain different regions which can never find out about each other's existence.

Answers to questions

1. Show that T and V are both proportional to χ^2.

Answer to question

We have:

$$T \propto v^2 = (Hr)^2 = (HR\chi)^2 = (HR)^2\chi^2$$

and

$$v \propto \frac{r^3}{r} = r^2 = R^2\chi^2$$

as required.

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2. Look up "quintessence" in a dictionary. Is it a well-chosen name?

Answer to question

My dictionary defines quintessence as "fifth substance, apart from the four elements, composing the heavenly bodies entirely & latent in all things". This is quite appropriate: dark energy does seem to be the major component of our universe, energetically speaking, and fills intergalactic space smoothly. In contrast our local environment (e.g. our galaxy) is dominated by the four substances baryonic matter, cold dark matter, neutrinos, and radiation, which perhaps are the modern equivalents of earth, water, air and fire.

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3. Calculate the present value of ρ_c assuming that $H_0 = 100$ km s^{-1} Mpc^{-1}.
Work out your answer in
(a) kg m^{-3},

(b) protons m^{-3},
(c) Galaxies per cubic Mpc (take 1 Galaxy = 10^{12} solar masses).

Answer to question

$$\rho_c(t_0) = \frac{3H_0^2}{8\pi G}.$$

(a) Now H_0 = 100 $\mathrm{kms^{-1}Mpc^{-1}}$ = 10^5 $\mathrm{ms^{-1}Mpc^{-1}}$. Our list of constants gives the length of a parsec in metres, which corresponds to 1 Mpc = 3.086×10^{22} m, so
$H_0 = (10^5/3.086\times10^{22})\ \mathrm{ms^{-1}m^{-1}} = 3.24\times10^{-18}\ \mathrm{s^{-1}}$
in SI units. Putting in the value of G from the constants list, we get
$\rho_c(t_0) = 1.878\times10^{-26}$ kg m^{-3}.
NB the units of G are N m^2 kg^{-2}. N is Newtons = kg m s^{-2}. This gives units of
s^{-2} from H^2 and m^3 $kg^{-1}s^{-2}$ from G, giving $\mathrm{kgm^{-3}}$ as advertised.

(b) A proton has mass 1.637×10^{-27} kg, from the constants list. To convert to protons per cubic metre, multiply by (protons per kilogram)= $1/m_p$, i.e

$\rho_c(t_0)$ = 11.47 protons m^{-3}.

(c) If a galaxy has mass 10^{12} solar masses, this is equal to 1.989×10^{42}kg, using the solar mass from the constants list. 1 cubic Mpc = $(3.086\times10^{22})^3$ m^3, so 1 galaxy $(\mathrm{Mpc})^{-3}$ is equal to 6.768×10^{-26} $\mathrm{kgm^{-3}}$, and

$\rho_c(t_0)$ = 0.3 galaxies $(\mathrm{Mpc})^{-3}$.

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4. Derive the above equation.

Answer to question

To get our equation for H, start with the Friedman Equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}$$

To find kc^2, substitute the first term on the right hand side with $H^2\Omega$, as derived on the previous line:

$$H^2 = H^2\Omega - \frac{kc^2}{R^2} \Rightarrow -\frac{kc^2}{R^2} = H^2 - H^2\Omega.$$

Tidying up a bit gives $-kc^2 = H^2R^2(1 - \Omega)$. This applies at any time, in particular, now:
$-kc^2 = H_0^2R_0^2(1 - \Omega_0)$.
So we can replace the second term on the right of the Friedman equation with

$$-\frac{kc^2}{R^2} = H_0^2(1-\Omega_0)\left(\frac{R_0}{R}\right)^2.$$

Finally, we replace the first term on the right with the long expression derived on the previous line, to give our final formula for the Hubble parameter at any given value of R, or alternatively of $R_0/R = (1 + z)$.

If you're worried by the previous equation, let's take it more slowly. Starting at the left, we have

$$H^2\Omega(t) = H^2\frac{\rho(t)}{\rho_c(t)} = \frac{H^2\rho(t)}{(3H^2/8\pi G)}.$$

Cancel H^2 on top and bottom:

$$H^2\Omega(t) = \frac{\rho(t)}{(3/8\pi G)} = \frac{8\pi G\rho(t)}{3}.$$

Second step in text is

$$\frac{8\pi G}{3} = \frac{H_0^2}{\rho_c(t_0)}$$

which is just the definition of ρ_c, applied at time t_0.

Third step is the same for each of the three terms in the bracket: we'll do it just for the first to cut down on the algebra. We have

$$H_0^2\frac{\rho_m(t)}{\rho_c(t_0)} = H_0^2\frac{\rho_m(t_0)(R_0/R)^3}{\rho_c(t_0)} = H_0^2\frac{\rho_m(t_0)}{\rho_c(t_0)}\left(\frac{R_0}{R}\right)^3.$$

Note that $\Omega_m = \rho_m(t_0)/\rho_c(t_0)$, the value for the present time, as it should be.

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5. Find the equation for R/R_0 when radiation dominates.

Answer to question

When radiation dominates we have $n = 4$, therefore

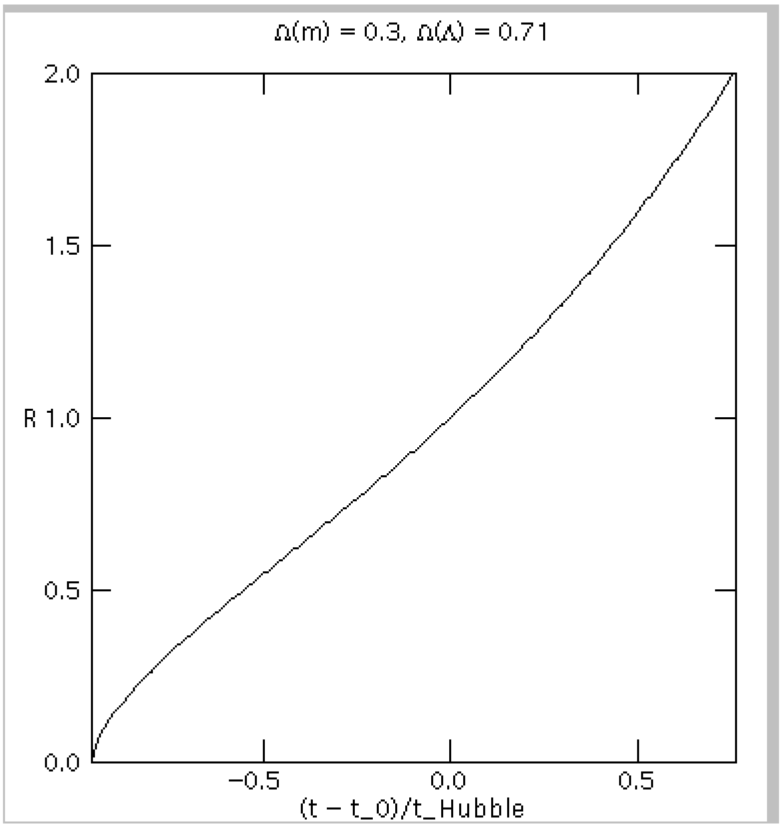
$$a = \frac{R}{R_0} = \left(\frac{4}{2}H_0 t\right)^{2/4} = \sqrt{2H_0 t}.$$

---

6. Current data suggest that $\Omega_\Lambda \approx 0.7$, $\Omega_m \approx 0.3$. Use the applet to plot $R(t)$ for this universe and hence read off the present age of the universe t_0 in terms of the Hubble time t_H.

Answer to question

Click on the left-hand panel of the applet roughly at $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$ (note that this is on the blue line $\Omega = 1$). The actual value you clicked on is reported in the title of the right-hand panel. If necessary re-click until you are pretty well there (due to the finite number of pixels you will get either 0.69 or 0.71 for Ω_Λ). You should get something like this:



It looks like $R = 0$ is reached just to the left of -1.0 on the bottom axis. If you want to be precise you could measure the distance from the edge of the plot to the tick at -0.5, and compare this to the gap between the 0.5 interval tick marks. In this way I find $R = 0$ is at -0.93, i.e. the start (time $t = 0$) is at:

$(t - t_0)/t_H = -0.93 \Rightarrow t_0 = 0.93 t_H$

In other words the present age of the Universe is the Hubble time to a pretty good approximation. Play around with the applet to convince yourself that this is not the case for most values of Ω !

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7. What is the condition on Ω_Λ for a Euclidean universe to recollapse? (Make reference to the applet).

Answer to question

Euclidean universes correspond to the blue line in the left-hand panel of the applet. The green line separates universes which re-collapse (on the left) from the ones which expand for ever (on the right). The blue and green lines cross at $\Omega_m = 1$, $\Omega_\Lambda = 0$, so Euclidean universes will recollapse if Ω_Λ is negative. (If it is zero we have the Einstein-de Sitter universe which expands for ever, as we have already seen).