1.1 Cosmological expansion and redshift

We can write the Hubble Law for the recession speed of a distant galaxy at distance r as:

$$v = Hr.$$

All three quantities in this equation change with time, so we will write them as functions, $r(t)$ and so on, when we want to emphasise the point. To be precise, $r(t)$ is the true or **proper distance**, that is, the distance that would be measured by a tape measure lying between us and the galaxy at time t. The recession speed v is the rate at which r is increasing, and H, the **Hubble parameter**, is constant everywhere in space because of homogeneity. Most theories predict that it changes with time, $H = H(t)$. In cosmology we give present-day values a subscript zero, so the Hubble parameter today, at time $t = t_0$, is $H(t_0) = H_0$, the so-called **Hubble constant**.

To see more clearly what is going on it is helpful to invent coordinates which expand along with the universe; we call these **co-moving coordinates**. If our galaxy has co-moving coordinate $x = 0$ and another galaxy has co-moving coordinate x, then the proper distance to it is

$$r = a(t)\, x$$

where $a(t)$ is called the **scale factor** and, like $H(t)$ depends on time but not position. If we put a dot over a symbol to denote the rate of change of the quantity, we can write the velocity v as

$$v = \dot{r} = Hr; \quad \Rightarrow \frac{d}{dt}(ax) = Hax\,.$$

But by definition, the co-moving coordinate is not affected by the Hubble expansion, so all the change in (ax) must belong to a. This means we can cancel x from our equation, giving

$$H(t) = \frac{\dot{a}(t)}{a(t)}\,.$$

Using this framework, we can separate the motion of real galaxies into two parts, the **Hubble flow** caused by the change of $a(t)$ and a so-called **peculiar motion** relative to the grid of co-moving coordinates (in effect, relative to the average motion of galaxies in a large surrounding region).

In an expanding universe, say how the following distances change with time:

	(a) increases	(b) stays the same	(c) decreases
The proper distance between two galaxies	☐	☐	☐
The comoving distance between two galaxies	☐	☐	☐
The proper distance across a galaxy	☐	☐	☐
The comoving distance across a galaxy	☐	☐	☐

Be very clear on the distinction between velocity and what is actually observed, i.e. the redshift, z, which is defined by

$$z \equiv \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}}.$$

Velocities are only inferred from the *approximate* formula

$$\frac{v}{c} \approx z\,.$$

Nowadays we routinely observe galaxies with redshift greater than one: are velocities *really* greater than the speed of light, despite what Einstein said? To answer this we need a deeper understanding of the redshift. Consider a photon setting out from one galaxy towards a nearby neighbour a proper distance dr away; nearby here means close enough that we can assume the small velocity difference between the galaxies dv is given by the simple formula $dv = cz$ (Fig. 1.1).

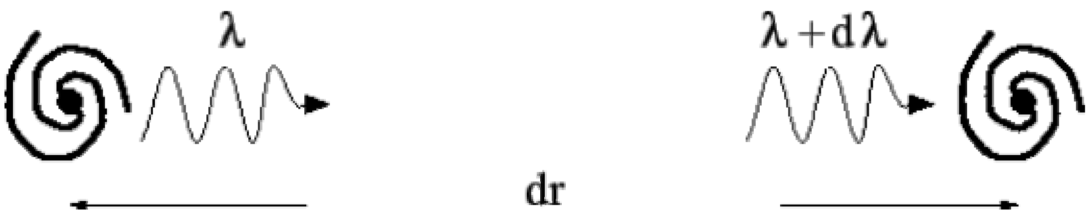

Figure 1.1: A photon travels between a pair of nearby galaxies

The photon makes the crossing in time $dt = dr/c$. During this time, the scale factor increases by a small amount $da = \dot{a}\ dt$. The relative speed of the galaxies is

$$dv = Hdr = \frac{\dot{a}}{a}cdt = c\frac{da}{a}\ .$$

Let's write the small wavelength shift $\lambda_{obs} - \lambda_{em} = d\lambda$; substituting this and the velocity-redshift formula, we get:

$$c\frac{da}{a} = dv = cz = c\frac{d\lambda}{\lambda}\ ,$$

that is, over the short time interval dt, the fractional change in the wavelength is the same as the fractional growth in the universe. Now suppose the photon passes the second galaxy and travels a long way through the universe before being detected in a third galaxy (ours). Exactly the same formula as above can be derived for every short segment of its path, so wavelength and scale factor will continue to track each other: $\lambda \propto a$, or

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_0)}{a(t_{em})}\,.$$

The photon expands at the same rate as the universe! The deep meaning of the **cosmological redshift**

$$z = \frac{\lambda_{obs}}{\lambda_{em}} - 1$$

is that $(1 + z)$ tells us how much the universe has expanded since the light was emitted.

Although we used the Doppler effect in the argument, the result of adding together all the infinitesimal Doppler shifts is very different from a single large Doppler shift. For instance, the redshift does not depend on the velocity $v(t)$, either now ($t0$), or at the time of emission ($t$em). To see this, suppose that the universe happened not to be expanding when the photon was emitted, ie. $H(t\text{em}) = 0$ and so $v(t\text{em}) = 0$. The universe then starts expanding, but stops again before the photon arrives, so also $v(t0) = 0$. The photon will still be redshifted because of the expansion that happened while it was *en route*.

All this tells us that the high redshifts of distant galaxies (the record is now at $z > 6$) do not necessarily imply recession speeds that are faster than light. As we suspected, the simple rule $cz = Hr$ is only true for small redshifts. But look again at the even simpler formula $v = Hr$, which we saw is always true, for *any* separation. If we go far enough away, *v will* become faster than the speed of light. Photons emitted by a galaxy at such a distance will never reach us, unless the universe starts to decelerate at some time in the future. We may or may not be able to see today photons that left this galaxy billions of years ago, depending on whether or not the universe has accelerated in the past (see Section 1.3.4).

Some cosmologists try to weasel out of this conclusion by claiming that the rate of increase of proper distance is not a "real" speed; they argue that it is the space between the galaxies which is expanding, while the galaxies are all more or less at rest. In effect this approach prefers co-moving distance to proper distance. It is sometimes useful to think this way, but there is no physical difference between two galaxies "really" moving apart and two galaxies stationary with the intervening space expanding; these are equivalent descriptions according to General Relativity (which gives a clue as to why the theory is so called).