GRAVITATION PC4771 2003 : Dr R.A. Battye

SPHERICALLY SYMMETRIC VACUUM SOLUTION

The general spherically symmetric metric is

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{1}$$

The corresponding curvature terms are

$$\begin{aligned}
R_{tt} &= \frac{1}{2}e^{\nu-\lambda}\left[\nu'' + \frac{1}{2}\nu'\left(\nu' - \lambda'\right) + \frac{2\nu'}{r}\right] + \dot{\lambda}\left(\dot{\nu} - \dot{\lambda}\right) - \frac{1}{2}\ddot{\lambda}\,,\\
R_{tr} &= \frac{\dot{\lambda}}{2r}\,,\\
R_{rr} &= \frac{1}{2}e^{\nu-\lambda}\left[\ddot{\lambda} - \frac{1}{2}\dot{\lambda}\left(\dot{\nu} - \dot{\lambda}\right)\right] - \frac{1}{2}\left[\nu'' + \frac{1}{2}\nu'\left(\nu' - \lambda'\right) - \frac{2\lambda'}{r}\right]\,,\\
R_{\theta\theta} &= 1 - e^{-\lambda}\left[1 + \frac{1}{2}r\left(\nu' - \lambda'\right)\right]\,,\\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}\,.
\end{aligned} \tag{2}$$

$R_{tr} = 0$ implies $\lambda = \lambda(r)$. Using $R_{tt} = R_{rr} = 0$ allows one to deduce that $\nu + \lambda = f(t)$. This constant of integration can be set to zero by allowing a reparameterization of the time coordinate. Hence, we deduce that $\nu = -\lambda$.

Therefore, $R_{\theta\theta} = 0$ implies

$$(r\,e^{\nu})' = 1 \quad \rightarrow \quad e^{\lambda} = 1 + \frac{C}{r}\,. \tag{3}$$

By comparison to the Newtonian spacetime

$$ds^2 \approx (1 + 2\varphi)dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \tag{4}$$

with $\varphi = -GM/r$ allows one to deduce that the constant $C = 2GM$.

Hence, we deduce the vacuum solution to the Einstein equations exterior to an object of mass M,

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,. \tag{5}$$

and

$$g_{\mu\nu} = \mathrm{diag}\left(1 - \frac{2GM}{r}, -\left(1 - \frac{2GM}{r}\right)^{-1}, -r^2, -r^2\sin^2\theta\right)\,. \tag{6}$$