**GRAVITATION PC4771 2003 : Dr R.A. Battye**

**ROTATING COORDINATE SYSTEMS**

Consider two plane polar coordinate systems $(t, r, \theta)$ and $(t', r', \theta') = (t, r, \theta - \omega t)$. One can compute the Jacobians,

$$J' = \begin{pmatrix} 1 & 0 & -\omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad J = \begin{pmatrix} 1 & 0 & \omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1}$$

The metric and its inverse are given by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -r^2 \end{pmatrix} , \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -r^{-2} \end{pmatrix} . \tag{2}$$

Hence, the transformed metric and its inverse can then by computed

$$g' = JgJ^T = \begin{pmatrix} 1 - r^2\omega^2 & 0 & -r^2\omega \\ 0 & -1 & 0 \\ -r^2\omega & 0 & -r^2 \end{pmatrix} , \quad g'^{-1} = J'^T g^{-1} J' = \begin{pmatrix} 1 & 0 & -\omega \\ 0 & -1 & 0 \\ -\omega & 0 & \omega^2 - r^{-2} \end{pmatrix} . \tag{3}$$

We are trying to compute the geodesic eqution for the coordinate $r'$, so we need to compute the corresponding Christoffel symbols. In particular, we would like to compute

$$\begin{aligned} \Gamma^r{}_{ij} &= \frac{1}{2} g^{rk} \left( -\partial_k g_{ij} + \partial_i g_{kj} + \partial_j g_{ik} \right) \\ &= \frac{1}{2} g^{rr} \left( -\partial_r g_{ij} + \partial_i g_{rj} + \partial_j g_{ir} \right) \\ &= -\frac{1}{2} g^{rr} \partial_r g_{ij} \, , \end{aligned} \tag{4}$$

where the metric now refers to that in the transformed coordinate system. Using the above expressions, it is simple to compute this Christoffel symbol

$$\Gamma^r{}_{ij} = \begin{pmatrix} -r\omega^2 & 0 & -r\omega \\ 0 & 0 & 0 \\ -r\omega & 0 & -r \end{pmatrix} . \tag{5}$$

The geodesic equation for $r'$ is then

$$\ddot{r}' + \Gamma^r{}_{ij} \dot{x}'^i \dot{x}'^j = 0 \, , \tag{6}$$

which implies that

$$\ddot{r} = r\omega^2 \dot{t}'^2 + 2r\omega \dot{t}' \dot{\theta}' + r\dot{\theta}'^2 , . \tag{7}$$

This can be rewritten as

$$\frac{\ddot{r}}{\dot{t}^2} = r\omega^2 + 2r\omega \frac{d\theta'}{dt'} + r \left( \frac{d\theta'}{dt'} \right)^2 . \tag{8}$$

The first term is the centrifugal force and the second is known as the coriollis force. These are the two 'pseudo' forces one experiences in a rotating frame of reference.