GRAVITATION PC4771 2003 : Dr R.A. Battye

PRELIMINARY AND REVISION MATERIAL

The aim of these notes is to provide you with sufficiently detailed introductory material in order for you to be able to understand the subsequent lectures. This material will be selectively discussed in the first lecture. The assumption is that you will have come across some of the material in sections 1.1, 1.2, 1.3 and 1.4 in previous courses. Sections 1.5 and 1.6 are 'new' and form the philosophical basis for General Relativity. If you are very unfamiliar with the material in sections 1.1 and 1.2 you should attend the special examples class on tensors and variational calculus which will be arranged in the first lecture.

1.1 Cartesian Tensors

1.1.1 Simple example : Conductivity

Here is a simple example which illustrates the concept of scalars, vectors and tensors. Consider Ohm's law for an isotropic medium

$$\mathbf{j} = \sigma \mathbf{E} \,. \tag{1}$$

Here, $\mathbf{j}$ is the current density **vector**, σ is the conductivity **scalar** and $\mathbf{E}$ is the electromagnetic field **vector**. In a more complicated medium (eg. a crystalline solid), Ohm's law takes the more general form

$$j_i = \sigma_{ij} E_j \,, \tag{2}$$

where we now consider the indices $i, j = 1, 2, 3$ and use the summation convention. $\sigma_{ij} = \sigma_{ji}$ is the symmetric conductivity **tensor** (or matrix). For isotropic systems $\sigma_{ij} = \sigma\delta_{ij}$ where δ_{ij} is the Kronecker delta, from which one can deduce (1)

1.1.2 Definitions

Consider two coordinate axes S and S' with respect to the same origin O with orthonormal basis vectors $\mathbf{e}_i$ in S and $\mathbf{e}'_i$ in S'. ie. $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ and $\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$.

(1) A scalar is a quantity which is invariant under coordinate transformation, for example between S and S'. If ϕ is its value in S and ϕ' is its value in S' then

$$\phi = \phi' \,. \tag{3}$$

(2) Now consider a vector $\mathbf{v}$ which has components v_i in S and v'_i in S' then

$$\mathbf{v} = v_i \mathbf{e}_i = v'_i \mathbf{e}'_i \quad \rightarrow \quad v'_i = L_{ij} v_j \,, \tag{4}$$

where $L_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$ is the orthogonal ($L_{ij} L_{kj} = \delta_{ik}$) rotation matrix representing the coordinate transformation. We define a vector to be an array of numbers whose values transform in this way under coordinate transformations.

(3) A Cartesian tensor of rank p in R^n is an array of n^p numbers which transform under rotations according to

$$T'_{i_1..i_p} = L_{i_1 j_1}..L_{i_p j_p} T_{j_1..j_p} \,. \tag{5}$$

Special cases : A scalar is a tensor of rank 0 and a vector is a tensor of rank 1. For second rank tensors

$$T'_{ij} = L_{ip} L_{jm} T_{pm} \,. \tag{6}$$

Example : If u_i and v_j are two vectors show that $T_{ij} = u_i v_j$ is a rank 2 tensor.

$$T'_{ij} = u'_i v'_j \quad u'_i = L_{ip} u_p \quad v'_j = L_{jm} v_m$$

$$\to T'_{ij} = L_{ip} L_{jm} u_p v_m = L_{ip} L_{jm} T_{pm} \,.$$

1.1.3 Properties of tensors

(1) Addition : one can add tensors of the same rank

$$A_{i_1..i_p} + B_{i_1..i_p} = C_{i_1..i_p} \,. \tag{7}$$

(2) Multiplication : rank(A)+rank(B)=rank(C)

$$A_{i_1..i_p} B_{j_1..j_q} = C_{i_1..i_p j_1..j_q} \,. \tag{8}$$

(3) Contraction : reduces the rank by 2

$$B_{i_3..i_p} = A_{i_1 i_1 i_3..i_p} \,. \tag{9}$$

(4) Symmetric-Antisymmetric split : one can write any tensor as the sum of a symmetric and antisymmetric part

$$A_{ij} = S_{ij} + T_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) + \frac{1}{2}\left(A_{ij} - A_{ji}\right) \,. \tag{10}$$

Moreover, the symmetric part can be written as a trace plus a trace-free part and in 3D antisymmetric tensors can be expressed in terms of a vector. The most general decomposition in 3D is

$$A_{ij} = \frac{1}{3} S \delta_{ij} + \left(S_{ij} - \frac{1}{3}\delta_{ij}\right) + \epsilon_{ijk} w_k \,, \tag{11}$$

where $w_k = \frac{1}{2}\epsilon_{ijk} T_{ij}$.

Note that : the trace of A is defined to be A_{ii} and ϵ_{ijk} is the total antisymmetric tensor of rank 3. Its value is +1 for symmetric permutations of $\{1, 2, 3\}$, -1 for antisymmetric perturbations and is 0 if any of i, j or k are the same.

1.1.4 Tensor detection theorem — Quotient rule

There exists a powerful theorem by which one can establish tensorial character. It has a general form which is tedious to write down. However, it is best understand by specific examples; their generalization being obvious. Therefore, consider the simple example: if $a_i = T_{ij}b_j$ is a vector for all arbitrary vectors b_j, then T_{ij} is a tensor.

Here, are a couple of variants. (1) if $a = T_i b_i$ is a scalar for all arbitrary vectors b_i then T_i is a vector. (2) if $a_{ij} = T_{ijkl}b_{kl}$ is rank 2 tensor for all rank 2 tensors b_{kl} then T_{ijkl} is a rank 4 tensor.

1.1.5 Isotropic tensors

Definition : An isotropic tensor is one whose components have the same value in any coordinate system. It is easy to show that δ_{ij} and ϵ_{ijk} are isotropic. It is more complicated,

yet possible, to establish that the only isotropic tensors of

rank 2 are $\lambda\delta_{ij}$,

rank 3 are $\lambda\epsilon_{ijk}$,

rank 4 are $\lambda\delta_{ij}\delta_{kl} + \alpha\delta_{ik}\delta_{jl} + \beta\delta_{il}\delta_{jk}$. where λ, α and β are constants

1.2 Variational Calculus

1.2.1 Euler-Lagrange equations

Let $L = L(x, y, y')$ where $y = y(x)$, $y' = dy/dx$ and define

$$I[y] = \int_{x=a}^{x=b} L(x, y, y')dx \,. \tag{12}$$

We seek paths $y(x)$ between fixed endpoints $y(a)$ and $y(b)$ which extremize $I[y]$. ie $\delta I =$ $I[y + \delta y] - I[y] = 0$ for all δy assumed to be small. We will use $\delta y(x) = \epsilon\eta(x)$ where $\eta(x)$ is arbitrary, subject to $\eta(a) = \eta(b) = 0$ and we shall ignore ϵ^2. Expanding using Taylor series one can deduce that

$$\begin{aligned} I[y + \epsilon\eta] &= \int_a^b L\left(x, y + \epsilon\eta, y' + \epsilon\eta'\right) dx \\ &= \int_a^b \left[L + \epsilon\eta\frac{\partial L}{\partial y} + \epsilon\eta'\frac{\partial L}{\partial y'}\right] dx + \mathcal{O}(\epsilon^2) \\ &= I[y] + \epsilon\int_a^b \left[\eta\frac{\partial L}{\partial y} + \eta'\frac{\partial L}{\partial y'}\right] dx + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Therefore, using integration by parts,

$$\delta I = I[y + \epsilon\eta] - I[y] = \epsilon\int_a^b dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\eta(x) + \mathcal{O}(\epsilon^2)\,. \tag{13}$$

We want this to be true for all η and hence we deduce the Euler-Lagrange equations

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial L}{\partial x} = \frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) . \tag{14}$$

1.2.2 First integrals

Solving the E-L can be difficult. Fortunately, if L is independent of x or y then first integrals exist.

(1) $\frac{\partial L}{\partial x} = 0$

$$\rightarrow L - y'\frac{\partial L}{\partial y'} = \mathrm{const} \tag{15}$$

(2) $\frac{\partial L}{\partial y} = 0$

$$\rightarrow \frac{\partial L}{\partial y'} = \mathrm{const} \tag{16}$$

1.2.3 Simple examples

(1) Geodesics on the unit sphere

The arc length on the unit sphere is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{17}$$

The distance between two fixed points θ_1 and θ_2 is given by

$$s = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} d\theta \left[1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]^{\frac{1}{2}} . \tag{18}$$

Therefore, one can use $L(\theta, \phi, \phi') = (1 + \sin^2\theta\phi'^2)^{\frac{1}{2}}$ to compute the trajectories between θ_1 and θ_2 which have the shortest distance. Such trajectories are called geodesics and will play a significant role in the subsequent development of General Relativity.

Since L is independent of ϕ one can deduce that

$$\frac{\phi'\sin^2\theta}{(1 + \sin^2\theta\,\phi'^2)^{\frac{1}{2}}} = \sin\alpha\,, \tag{19}$$

where α is the minimum value of θ which occurs at $\phi = \phi_0$. One can rearrange and integrate to give

$$\phi = \phi_0 + \int_\alpha^\theta d\theta \frac{\sin\alpha}{\sin\theta\sqrt{\sin^2\theta - \sin^2\alpha}} . \tag{20}$$

I leave it as a non-trivial exercise for you to show that

$$\tan\theta\cos(\phi - \phi_0) = \tan\alpha\,. \tag{21}$$

(2) Soap Films

Consider the surface of revolution of $y = y(x)$ between two points (x_1, y_1) and (x_2, y_2).

The area of an infinitesimal surface element $dA = 2\pi y ds$ where s is the arc length. Therefore, the total surface area is given by

$$A = 2\pi \int_{x_1}^{x_2} dx\, y \left(1 + y'^2\right)^{1/2} . \tag{22}$$

Note that the volume enclosed is given by

$$V = \pi \int_{x_1}^{x_2} dx\, y^2 . \tag{23}$$

One can extremize the surface area by considering

$$L(x, y, y') = y \left(1 + y'^2\right)^{1/2} . \tag{24}$$

Since L is independent of x, there is a first integral

$$L - y' \frac{\partial L}{\partial y'} = y \left(1 + y'^2\right)^{1/2} - \frac{y'^2 y}{(1 + y'^2)^{1/2}} = c_1 , \tag{25}$$

from which one can deduce that

$$\frac{dx}{dy} = \frac{c_1}{\sqrt{y^2 - c_1^2}} \quad \rightarrow \quad y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right) . \tag{26}$$

1.2.4 Generalizations

(1) $L = L(x_i, y_j, \partial y_j/\partial x_i)$

$$\frac{d}{dx_i}\left(\frac{\partial L}{\partial(\partial y_j/\partial x_i)}\right) - \frac{\partial L}{\partial y_j} = 0 \tag{27}$$

(2) $L = L(x, y, y', y')$

$$\frac{d^2}{dx^2}\left(\frac{\partial L}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) + \frac{\partial L}{\partial y} = 0 \tag{28}$$

(3) If $K = f(L)$ then K is satisfies the E-L equation if L does.

$$\frac{d}{dx}\left(\frac{\partial K}{\partial y'}\right) - \frac{\partial K}{\partial y} = \frac{\partial f}{\partial L}\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right] = 0 \tag{29}$$

1.3 Newtonian mechanics and gravity

This material should be very familiar, but we will formulate it in a slightly different way so as to facilitate the motivation of General Relativity in the subsequent sections.

1.3.1 Newton's laws

(1) A particle will continue in uniform motion unless acted upon by a force.

(2) $\mathbf{F} = m_{\rm i}\mathbf{a}$ where $m_{\rm i}$ is the inertial mass, $\mathbf{F}$ is the force and $\mathbf{a}$ is the acceleration.

(3) Action and reaction are equal and opposite.

1.3.2 Inertial frames

Newton's 1st law can only be true for a special class of reference frames. If it is true for a particular reference frame, then it will be true for all other in uniform (non-rotational) motion relative to it. This special class of frames are known as **inertial frames**. The acceleration of the 2nd law must be defined relative to an inertial frame.

1.3.3 Galilean Relativity

Newton's 2nd law is invariant under Galilean transformations to another inertial frame:
If S and S' are two frames moving relative to each other with velocity $\mathbf{v}$, then the position in S, $\mathbf{x}$, and that in S', $\mathbf{x}'$, are related by $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$. Hence, one can deduce that $\mathbf{a} = \mathbf{a}'$.

1.3.4 Newtonian gravity

Force on body 1 at $\mathbf{x}_1$ due to body 2 at $\mathbf{x}_2$ is

$$\mathbf{F}_{12} = G m_{1\rm gp} m_{2\rm ga} \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3} \propto \frac{1}{|\mathbf{x}_2 - \mathbf{x}_1|^2}\,, \tag{30}$$

where $m_{1\rm gp}$ is the passive gravitational mass of body 1 of $m_{2\rm ga}$ is the active gravitational mass of body 2.

Newton's 3rd law implies $\mathbf{F}_{12} = -\mathbf{F}_{21}$ from which one can deduce that $m_{1\rm gp}m_{2\rm ga} = m_{2\rm gp}m_{1\rm ga}$. Therefore, if Newton's 3rd law holds then the ratio of active and passive gravitational masses is universal. By choice of units let this ratio be unity and let us now just consider the gravitational mass $m_{\rm g}$.

The gravitational force can be derived from a potential ϕ_2 due to 'body' 2

$$F_{12} = -m_{\rm g}\nabla\phi_2\,, \tag{31}$$

which satisfies the Poisson equation

$$\nabla^2\phi_2 = -4\pi G\rho_2(\mathbf{x})\,. \tag{32}$$

This can be solved by standard potential theory

$$\phi_2(\mathbf{x}) = -G\int d^3\mathbf{x}'\,\frac{\rho_2(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,, \tag{33}$$

which can be expressed in the familiar form $\phi_2 = -Gm_{2\rm g}/r$ if the density distribution is that of a point particle : $\rho_2(\mathbf{x}) = M_2\delta(\mathbf{x})$.

1.3.5 Weak equivalence principle (WEP)

Assertion : the ratio of the inertial and gravitational masses of bodies is a universal constant. Without loss of generality (wlog) we choose this to be unity.

This is often known as Galileo's law or principle since it is a consequence of his famous experiment conducted at the leaning tower of Pisa.

Its veracity can be probed using a pendulum as first used by Eotvos in 1889. He measured the angle, θ, that a static pendulum adopted relative to some fixed reference direction, such as the centre of the earth. This is established by a balance between the centrifugal force ($\propto m_{\rm i}$) due to the rotation of the earth and the gravitational force ($\propto m_{\rm g}$), enabling the ratio to be measured. In 1889 the WEP was found to be correct to 1 part in 5000. A hundred years later this had been improved to 1 part in 10^{13}!

The WEP implies that it is impossible to disentangle the effects of a gravitational field from those due to a uniformly accelerating frame using the motion of test particles. Note that this would be possible for the electromagnetic field since one could use particles with different ratios of mass to charge.

1.3.6 Particle orbits

Consider a particle of mass m moving in 2D around a another with mass $M(>> m)$ assumed to be fixed. If the position of the particle is $\mathbf{x} = r\hat{\mathbf{r}}$ where $\hat{\mathbf{r}} = (\cos\phi, \sin\phi)$ then

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\mathbf{p}} \\ \ddot{\mathbf{x}} &= = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\mathbf{p}}\,, \end{aligned} \tag{34}$$

where $\hat{\mathbf{p}} = (-\sin\phi, \cos\phi)$. The gravitational force is in the direction of $\hat{\mathbf{r}}$ so one can deduce that

$$r^2\dot{\phi} = \frac{h}{m}\,, \tag{35}$$

and

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}\,. \tag{36}$$

By substituting (35) into (36) can then integrating we establish that

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} - \frac{h^2}{2m^2r^2} - E\,, \tag{37}$$

where E is a constant which corresponds to the energy of the system.

If one defines $f(r)$ such that

$$\frac{1}{2}\dot{r}^2 = f(r)\,, \tag{38}$$

then there exist a circular orbit with radius $r_\star$ when $f(r_\star) = 0$ and $f'(r_\star) = 0$. The orbit is stable if $f''(r_\star) < 0$.

Using (37) one can deduce that there exists a stable circular orbit with

$$r_\star = \frac{h^2}{GMm^2} \quad \text{and} \quad E = \frac{1}{2}\left(\frac{GMm}{h}\right)^2 . \tag{39}$$

Of course the planetary orbits are not circular. In order to study more general orbits it is conventional to define $u = 1/r$, and hence

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} , \tag{40}$$

which has a solution

$$u = \frac{GM}{h^2} + C\cos(\phi - \phi_0) , \tag{41}$$

where C and ϕ_0 are constants of integration. If one defines $\ell^{-1} = GM/h^2$ and $e = C\ell$ then

$$\frac{\ell}{r} = 1 + e\cos(\phi - \phi_0) , \tag{42}$$

which implies that the orbits are conic sections with semi-latus rectum ℓ and eccentricity e. If $0 < e < 1$ this is an ellipse and if $e = 0$ it is a circular orbit.

1.4 Review of Special Relativity

1.4.1 Principles of Special Relativity

(1) all inertial observers are equivalent

$\rightarrow$ the laws of physics are the same for all inertial observers

$\rightarrow$ the Principle of General covariance : the laws of physics are frame independent. If $F(A, B, C, ..) = 0$ is some physical law connecting observables $A, B, C, ..$ in frame S then $F(A', B', C', ..) = 0$ in S' where $A', B', C', ..$ are the transformed observables.

(2) the velocity of light is the same in all inertial systems

$\rightarrow$ the Galilean law of relative velocities cannot apply.

Comments: (i) there exist local inertial frames (LIFs) with respect to which light travels in a straight line. (ii) space and time are not absolute but are related by Lorentz transformations.

1.4.2 Minkowski space-time

Special relativity is formulated in terms of 'flat' or Minkowski space time. If $P(x^\mu)$ and $Q(x^\mu + dx^\mu)$ are two events then the interval between P and Q is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu , \tag{43}$$

where we have used the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and the position 4-vector $X^\mu = (t, x^i)$. $|ds|^{1/2} = d\tau$ is the proper time or distance between two events.

Timelike trajectories on which observers exist have $ds^2 < 0$, light (and other massless particles) travel on null trajectories with $ds^2 = 0$, and hypothetical particles, known as tachyons, which travel faster than the speed of light exists on spacelike trajectories with $ds^2 > 0$.

In Cartesian spaces the distance between two points $\mathbf{dx}$ is invariant under coordinate rotations. ie $\mathbf{dx}' = L\mathbf{dx}$ for an orthogonal rotation matrix L. Minkowski space-time is invariant under coordinate rotations and also the translational symmetries of R^3. However, one must be able to relate frames which are moving at a velocity v relative to each other.

This is done using the Lorentz transformation (LT) or rotation which acts to rotate coordinates between time and space. In terms of the position 4-vector this is usually expressed as

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu} \,, \tag{44}$$

where $\Lambda^{\mu}{}_{\nu}\, \Lambda^{\nu}{}_{\rho} = \delta^{\mu}{}_{\rho}$.

The dot product of two 4-vectors A^{μ} and B^{ν} given by $A \cdot B = \eta_{\mu\nu} A^{\mu} B^{\nu}$ is a Lorentz invariant in the same way as the length of a vector is invariant under spatial rotations. ie it is invariant under the Lorentz transformation and its value remains unchanged under Lorentz boosts. A simple consequence of this is that the interval ds^2 is a Lorentz invariant.

For a 2D Minkowski space-time

$$\begin{pmatrix} dt' \\ i\,dx' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dt \\ i\,dx \end{pmatrix} = \gamma_v \begin{pmatrix} 1 & i\,v \\ -i\,v & 1 \end{pmatrix} \begin{pmatrix} dt \\ i\,dx \end{pmatrix} \,, \tag{45}$$

where $\gamma_v = (1 - v^2)^{-1/2}$ and $\sin\theta = i\,v\gamma_v$.

The Lorentz transformations form a group:

$$\Lambda^{\mu}{}_{\nu}(0) = \delta^{\mu}{}_{\nu} \,, \quad \Lambda^{\mu}{}_{\nu}(-v) = [\Lambda^{\mu}{}_{\nu}]^{-1}(v) \,, \quad \Lambda^{\mu}{}_{\nu}(v_1)\Lambda^{\nu}{}_{\rho}(v_2) = \Lambda^{\mu}{}_{\rho}(v_1 + v_2) \,. \tag{46}$$

The full group of symmetries of Minkowski space-time is known as the Poincare group which includes the LTs, spatial rotations and translational invariance along with time translation invariance.

1.4.3 Doppler-shift formula

Consider two clocks at rest at the origin of 2D inertial frames S and S' whose relative velocity is v. The clocks indicate $t = t' = 0$ when the spatial origins coincide. When the clock in S' reads $\Delta t'$ it receives a signal from the clock in S sent at time Δt.

The signal is emitted at $(t, x) = (\Delta t, 0)$ in S which using the LT is $(t', x') = \gamma_v(\Delta t, -v\Delta t)$ in S'. The signal is received at $(t', x') = (\Delta t', 0)$ The time as observed in S' is the sum of the emission time and the time taken to get to $x' = 0$. Hence

$$\Delta t' = \gamma_v \Delta t + v\gamma_v \Delta t = \left(\frac{1 + v}{1 - v}\right)^{1/2} \Delta t \,, \tag{47}$$

from which one can deduce the frequency/wavelength shift of

$$\frac{\lambda'}{\lambda} = \frac{\nu}{\nu'} = \frac{\Delta t'}{\Delta t} = \left(\frac{1+v}{1-v}\right)^{1/2} = 1 + v + \mathcal{O}(v^2)\,. \tag{48}$$

1.4.4 4-velocity and the addition formula

The 4-velocity U^μ is defined by

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right) = \frac{dt}{d\tau}\left(1, \mathbf{v}\right)\,, \tag{49}$$

where $\mathbf{v} = d\mathbf{x}/d\tau$ and $dt/d\tau = (1-v^2)^{-1/2}$. If the velocity of light is constant for all inertial observers implies that the standard relative velocity law of addition cannot hold in Special Relativity.

Consider two particles moving at velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to the origin. Hence, the 4-velocities are given by

$$U_1^\mu = \gamma_{v_1}(1, \mathbf{v}_1)\,, \quad U_2^\mu = \gamma_{v_2}(1, \mathbf{v}_2)\,. \tag{50}$$

If we transform to a frame where particle 1 is stationary

$$U_1^\mu = (1, \mathbf{0})\,, \quad U_2^\mu = \gamma_v(1, \mathbf{v})\,, \tag{51}$$

where v is the relative velocity. Since $U_1 \cdot U_2$ is a Lorentz invariant

$$U_1 \cdot U_2 = \eta_{\mu\nu} U_1^\mu U_2^\nu = \gamma_v = \gamma_{v_1}\gamma_{v_2}\left(1 - \mathbf{v}_1 \cdot \mathbf{v}_2\right)\,, \tag{52}$$

from which one can deduce that

$$v^2 = \frac{\left(\mathbf{v}_1 - \mathbf{v}_2\right)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2}\,. \tag{53}$$

If $v_1, v_2 << 1$ then

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2\,, \tag{54}$$

and if either v_1 or v_2 equal one then $v = 1$.

1.4.5 4-momentum

The 4-momentum is defined to be

$$P^\mu = mU^\mu = (m\gamma_v, m\gamma_v\mathbf{v}) = (E, \mathbf{p})\,, \tag{55}$$

where m is the rest mass, E is the energy and $\mathbf{p}$ is the 3-momentum. The quantity

$$\eta_{\mu\nu}P^\mu P^\nu = E^2 - |\mathbf{p}|^2\,, \tag{56}$$

is a Lorentz invariant and therefore can be equated with its value in the rest frame, m^2. Hence $E^2 = |\mathbf{p}|^2 + m^2$. Also the 4-momentum can be transformed using the LT

$$\begin{pmatrix} E' \\ p' \end{pmatrix} = \gamma_v \begin{pmatrix} 1 & v \\ -v & 1 \end{pmatrix} . \tag{57}$$

If $m = 0$ then $E = |\mathbf{p}| = hf = \hbar\nu$ and hence the 4-momentum of a photon can be written as

$$P^\mu = \hbar\nu \left(1, \hat{\mathbf{k}}\right) , \tag{58}$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of motion. One can always move to a frame where the total 3-momentum is zero. Such a frame is called the centre of momentum frame.

In an elastic collision the total 4-momentum is conserved. ie

$$P^\mu_{\rm tot}(\text{in}) = P^\mu_{\rm tot}(\text{out}) . \tag{59}$$

These concepts are best understood by reference to examples :

(1) A body of rest mass m_0 disintegrates into two parts with rest masses m_1 and m_2. Compute the energies E_1 and E_2 of the decay products.

Before : $P^\mu = (m_0, \mathbf{0})$

After : $P^\mu_1 = (E_1, \mathbf{p}_1)$ and $P^\mu_2 = (E_2, \mathbf{p}_2)$

Conservation of 4-momentum implies that

$$m_0 = E_1 + E_2 \quad \text{and} \quad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0} . \tag{60}$$

Also $E_1^2 = |\mathbf{p}_1|^2 + m^2$ and $E_2^2 = |\mathbf{p}_2|^2 + m^2$. Hence, one can deduce that

$$E_1 = \frac{m_0^2 + m_1^2 - m_2^2}{2m_0} , \quad E_2 = \frac{m_0^2 - m_1^2 + m_2^2}{2m_0} . \tag{61}$$

(2) An unstable pion π_0 of rest mass m_π and velocity v can decay into two photons via the process

$$\pi_0 \to \gamma + \gamma , \tag{62}$$

Compute the angle θ between the two photons as measured in an inertial laboratory frame in terms of m_π and the frequencies of the two photons ν_1 and ν_2.

Before : $P^\mu = (m_\pi, \gamma m_\pi \mathbf{v})$

After : $P^\mu_1 = \hbar\nu_1(1, \hat{\mathbf{k}}_1)$ and $P^\mu_2 = \hbar\nu_2(1, \hat{\mathbf{k}}_2)$ with $\cos\theta = \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2$.

Conservation of 4-momentum implies that

$$\gamma m_\pi = \hbar(\nu_1 + \nu_2) \quad \text{and} \quad \gamma m_\pi \mathbf{v} = \hbar(\nu_1 \hat{\mathbf{k}}_1 + \nu_2 \hat{\mathbf{k}}_2) . \tag{63}$$

Squaring and subtracting the two equations gives

$$2\hbar^2\nu_1\nu_2(1-\cos\theta) = m_\pi^2\,, \tag{64}$$

and hence

$$\sin\frac{\theta}{2} = \frac{m_\pi}{2\hbar\sqrt{\nu_1\nu_2}}\,. \tag{65}$$

1.4.6 Maxwell's laws of electromagnetism

Using units where $\epsilon_0 = 1$ and $\mu_0 = 1$, the laws of electromagnetism are

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \rho \qquad & \nabla\times\mathbf{E} &= \mathbf{j} + \frac{\partial\mathbf{E}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 \qquad & \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 \end{aligned} \tag{66}$$

from which one can deduce the continuity relation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0\,. \tag{67}$$

In order to adapt these laws into Special Relativity define the field strength tensor

$$F^{\mu\nu} = -F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{68}$$

and the current density 4-vector

$$j^\mu = (\rho, \mathbf{j})\,. \tag{69}$$

Then one can write Maxwell's equations as

$$\partial_\mu F^{\mu\nu} = j^\nu\,, \quad \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0\,, \tag{70}$$

where

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)\,, \tag{71}$$

and the continuity equation becomes $\partial_\mu j^\mu = 0$. Now consider the potential formulation of electromagnetism in terms of the scalar potential, ϕ, and the vector potential, $\mathbf{A}$, using this

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\,, \quad \mathbf{B} = \nabla\times\mathbf{A}\,. \tag{72}$$

If we define $A^\mu = (\phi, \mathbf{A})$ then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{73}$$

which is invariant under the gauge transformations $A_\mu \to A_\mu + \partial_\mu \psi$. The gauge can be fixed using, for example, the Lorentz gauge $\partial_\mu A^\mu = 0$.

1.5 Einstein's lift experiment

1.5.1 Strong equivalence principle

Consider a particle moving under the effects of gravity and some external force $\mathbf{F}$. Newton's 2nd law implies that

$$m_{\rm i}\mathbf{a} = m_{\rm g}\mathbf{g} + \mathbf{F}\,. \tag{74}$$

If the WEP holds then $m_{\rm i} = m_{\rm g}$ and

$$m_{\rm i}\left(\mathbf{a} - \mathbf{g}\right) = \mathbf{F}\,. \tag{75}$$

By observing that there exists freely falling frames (FFF) where $\mathbf{a} = \mathbf{g}$, Einstein asserted the Strong equivalence principle:

There are no local experiments which can distinguish non-rotating free-fall in a gravitational field from uniform motion in the absence of a gravitational field.

NB. FFFs can only be defined over regions where $\mathbf{g}$ is constant. ie. they can only be defined locally.

1.5.2 Simple thought experiment

Consider a lift in which there is an observer O and two lights one on the ceiling of the lift L_1, a distance $dl << 1$ above O, and one on the side of the lift L_2. Also consider another observer on the ground at O' and assume that in the vicinity of the lift $\mathbf{g}$.

The time taken for a photon emitted by L_1 to get to reach O is $dt = dl$ (NB we are using c=1).

Now cut the wire!

1.5.3 Gravitational redshift

The lift is now a FFF. If the SEP holds then it is also a LIF which implies that SR holds in the lift and the observer O sees no frequency change in the light emitted by L_1.

But O' sees the lift moving at a speed $du = gdt = gdl$. Using the Doppler shift formula with $v = -du$ one finds that

$$\frac{d\nu}{\nu} = du = gdl\,. \tag{76}$$

If $g = -\frac{\partial \phi}{\partial l}$ then

$$\frac{d\nu}{\nu} = -dl\frac{\partial \phi}{\partial l} = -d\phi\,. \tag{77}$$

This is the gravitational redshift formula which relates the change frequency as seen by O' to the change in gravitational potential ϕ.

1.5.4 Fermat's principle and the SEP

Now consider the emitted horizontally by L_2. Since the lift is an LIF the coordinates in the lift are $(x_l, y_l) = (t, d)$ where d is a constant. Is we assume that the motion of the lift is non-relativistic then the coordinates relative to the ground are

$$(x_g, y_g) = \left(x_c, y_c - \frac{1}{2}gt^2\right) = \left(t, d - \frac{1}{2}gt^2\right) . \tag{78}$$

Hence, as viewed from O' the photon has a curved path. Fermat's principle asserts that light takes the shortest path between two points. If the speed of light is constant this corresponds to extremizing the distance between the points

$$s = \int_{x_1}^{x_2} ds . \tag{79}$$

Such trajectories are known as geodesics. In Minkowski space-time the geodesics are straight lines. The argument above suggests that light does not travel in a straight lines when in free-fall if the SEP holds. Hence, the assertion that the SEP holds requires one to either drop the cherished notion of Fermat's principle or the idea that the space-time is flat. This realization motivates the formulation of General Relativity to study the effects of gravity in a relativistic context.

1.6 Einstein's vision of General Relativity

Faced with this choice Einstein's vision was to realise that space must be curved. His ideas can be summarized as

1. Space-time is a pseudo-Riemannian manifold (we shall define these terms later) with a metric which depends on position. ie

$$ds^2 = g_{\mu\nu}(x^\mu)\, dx^\mu dx^\nu . \tag{80}$$

We can always choose coordinates so that space is locally flat and the geodesics are straight. This allows us to classify them as time-like ($ds^2 > 0$), null ($ds^2 = 0$) and space-like ($ds^2 < 0$).

2. Null geodesics represent light rays. Timelike geodesics represent the world lines of freely falling particles and along these lines the arc length measures the time shown by standard clocks.

3. Energy-momentum curves space-time. This has some similarities to another concept known as Mach's principle.