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SCHWARZSCHILD SOLUTION IN A NUTSHELL

Most general spherically symmetric metric :

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{1}$$

Spherically symmetric vacuum solution :

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{2}$$

The Schwarzschild radius is $r_{\rm S} = 2GM$ and M is known as the Schwarzschild mass.

Birkhoff's theorem :

Spherically symmetric vacuum solution exterior to a mass distribution must be static and the form of the metric is that of the Schwarzschild solution.

Dynamics in the Schwarzshild solution :

$$L_{\rm eff} = \left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) . \tag{3}$$

If $\theta \equiv \pi/2$, $E = (1 - r_{\rm S}/r)\dot{t}$ and $r^2\dot{\phi} = h$ then

$$\dot{r}^2 = E^2 - V(r) , \tag{4}$$

where

$$V(r) = \left(L + \frac{h^2}{r^2}\right)\left(1 - \frac{r_{\rm S}}{r}\right) , \tag{5}$$

and $L = 0$ for photons, $L = 1$ for timelike particles and $L = -1$ for spacelike particles.

Gravitational redshift :

$$\frac{\nu_1}{\nu_2} = \sqrt{\frac{g_{tt}(r_2)}{g_{tt}(r_1)}} \approx 1 - \varphi_2 + \varphi_2 . \tag{6}$$

Light deflection :

If $u = r_0/r$, $r_0 = h/E$ and $\epsilon = r_{\rm S}/r_0$ then

$$\left(\frac{du}{d\phi}\right)^2 = 1 - u^2 + \epsilon u^3 , \tag{7}$$

and

$$u(\phi) = \sin\phi + \epsilon\left[A\cos\phi + \frac{1}{2}\left(1 + \cos^2\phi\right)\right] + \mathcal{O}(\epsilon^2) , \tag{8}$$

from which we can deduce that light deflection is $\delta = 4GM/r_0$ by setting $u = 0$ at $\phi = \pi + \phi_2$ and $\phi = -\phi_1$ with $\delta = \phi_1 + \phi_2$.

Perihelion precession :

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2 - 1}{h^2} - u^2 + \frac{r_\mathrm{S}}{h^2}u + r_\mathrm{S}u^3\,, \tag{9}$$

where $u = 1/r$. To lowest order there is a solution

$$\frac{l}{r} = 1 + e\cos\left[\omega(\phi - \phi_0)\right]\,, \tag{10}$$

where

$$\omega = 1 - \frac{3r_\mathrm{S}^2}{4h^2} + \mathcal{O}\left(r_\mathrm{S}^4/h^4\right)\,. \tag{11}$$

This leads to a shift in the perhelion of

$$\frac{3\pi r_\mathrm{S}^2}{2h^2}\,. \tag{12}$$

Black holes :

$$dr_\star = \left(1 - \frac{2GM}{r}\right)^{-1} dr \quad \to \quad r_\star = r + 2GM\log\left|\frac{r}{2GM} - 1\right|. \tag{13}$$

If we define the Eddington-Finkelstein coordinates, $u = t - r_\star$ and $v = t + r_\star$ then

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dudv - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{14}$$

Light cones flip by 90° as one goes through the event horizon $r = 2GM$.