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THE EINSTEIN EQUATIONS IN A NUTSHELL

Energy-momentum tensor of a perfect fluid :

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu - Pg^{\mu\nu} \quad \text{and} \quad \nabla_\mu T^{\mu\nu} = 0\,.$$

Energy conditions :

$$\text{Weak}: \ \rho > 0\,, \ \ \rho + P > 0\,. \qquad \text{Strong}: \ \rho + 3P > 0\,.$$

SEP to geodesic equation :

$$\text{SEP}: \quad \frac{d^2x^\mu}{d\tau^2} = 0\,, \quad \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \ \text{(light)} \ \text{or} \ 1 \ \text{(particles)}\,.$$

$$\text{Transformation}: \ x'^\mu \ \rightarrow \ dx^\mu = J^\mu{}_\nu dx'^\nu \ \rightarrow \ g_{\mu\nu}\frac{dx'^\mu}{d\tau}\frac{dx'^\nu}{d\tau}g_{\mu\nu} = 0 \ \text{or}\,, 1$$

$$\text{and} \quad \frac{d^2x'^\alpha}{d\tau^2} + \Gamma'^\alpha{}_{\gamma\nu}\frac{dx'^\gamma}{d\tau}\frac{dx'^\nu}{d\tau} = 0\,,$$

$$\text{where} \quad g_{\mu\nu} = J^\alpha{}_\mu J^\beta{}_\nu \eta_{\alpha\beta}\,, \qquad \Gamma'^\alpha{}_{\gamma\nu} = J'^\alpha{}_\beta J^\rho{}_\gamma \partial_\rho J^\mu{}_\nu\,.$$

Einstein equations :

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,, \quad \text{or} \quad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\,.$$

Newtonian limit :

$$ds^2 = e^{-2\varphi}dt^2 - dx^2 - dy^2 - dz^2\,, \quad \rightarrow \quad \ddot{x}^i - \partial^i\varphi\, e^{-2\varphi}\dot{t}^2 = 0\,, \quad \frac{d^2\mathbf{x}}{d\tau^2} = -\nabla\varphi + \mathcal{O}(\varphi^3)\,.$$

$$\rightarrow \Gamma^i{}_{tt} = -\partial^i\varphi e^{-2\varphi}\,, \quad \Gamma = 0 \quad \text{otherwise}\,, \quad \text{and} \quad R_{tt} = \nabla^2\varphi$$

$$\nabla^2\varphi = 4\pi G\rho\,.$$

Gravitational Radiation :

$$\partial_\rho\partial^\rho h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right)\,,$$

$$\text{in a gauge where} \quad \partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0\,.$$