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CURVATURE IN A NUTSHELL

Derviatives commute for a scalar :

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\phi = 0\,. \tag{1}$$

Ricci Identity :

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,A^\rho = R^\rho{}_{\alpha\mu\nu}\,, \quad (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,A_\rho = R^\alpha{}_{\rho\nu\mu} A_\alpha\,. \tag{2}$$

$$R^\rho{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\rho{}_{\alpha\nu} - \partial_\nu \Gamma^\rho{}_{\alpha\mu} + \Gamma^\rho{}_{\gamma\mu}\Gamma^\gamma{}_{\alpha\nu} - \Gamma^\rho{}_{\gamma\nu}\Gamma^\gamma{}_{\alpha\mu}\,. \tag{3}$$

Ricci tensor and scalar

$$R_{\beta\alpha\mu\nu} = g_{\beta\rho} R^\rho{}_{\alpha\mu\nu}\,, \quad R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}\,, \quad R = R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta}\,. \tag{4}$$

In a locally inertial frame :

$$R_{\gamma\alpha\mu\nu} = \frac{1}{2}\left[\partial_\mu\partial_\alpha g_{\gamma\nu} + \partial_\nu\partial_\gamma g_{\alpha\mu} - \partial_\mu\partial_\gamma g_{\alpha\nu} - \partial_\nu\partial_\alpha g_{\gamma\mu}\right]\,. \tag{5}$$

One can deduce the following symmetry relations which are true in all frames since they are tensorial

$$R_{\gamma\alpha\mu\nu} = -R_{\alpha\gamma\mu\nu} = -R_{\gamma\alpha\nu\mu} = R_{\mu\nu\gamma\alpha}\,, \tag{6}$$

$$R_{\gamma\alpha\mu\nu} + R_{\gamma\nu\alpha\mu} + R_{\gamma\mu\nu\alpha} = 3R_{\gamma[\alpha\mu\nu]} = 0 \tag{7}$$

Bianchi identity :

$$\nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\mu R_{\gamma\alpha\nu\beta} + \nabla_\nu R_{\gamma\alpha\beta\mu} = \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\mu R_{\nu\mu\gamma\alpha} + \nabla_\nu R_{\beta\mu\gamma\alpha} = 3\nabla_{[\beta} R_{\mu\nu]\gamma\alpha} = 0\,. \tag{8}$$

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0\,. \tag{9}$$

Round trips by parallel transport :

Consider a vector A^μ parallelly transported around a small, closed curve, the change in the vector is given by

$$\Delta A^\mu = -\frac{1}{2} R^\mu{}_{\alpha\rho\beta} \int x^\beta \frac{dx^\rho}{du} du\,. \tag{10}$$

Geodesic deviation :

$$\ddot{\delta x}^\alpha + R^\alpha{}_{\beta\gamma\sigma} T^\beta T^\gamma \delta x^\sigma = 0\,. \tag{11}$$