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CONNECTION AND TENSOR CALCULUS IN A NUTSHELL

Partial differentiation of a vector is not a tensor :

$$\begin{aligned} \partial'_\mu A'_\nu &= J^\alpha{}_\mu J^\beta{}_\nu \partial_\alpha A_\beta + J^\alpha{}_\mu A^\beta \partial_\alpha J^\beta{}_\nu \,, \\ \partial'_\mu A'^\nu &= J^\alpha{}_\mu J'^\nu_\beta \partial_\alpha A^\beta + J^\alpha{}_\mu A^\beta \partial_\alpha J'^\nu_\beta \,. \end{aligned} \tag{1}$$

Parallel transport via connection : $\bar{\delta} A^\nu = -\Gamma^\nu{}_{\alpha\beta}\delta x^\alpha A^\beta$.

Covariant differentiation :

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\beta} A^\beta \,, \quad \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha{}_{\mu\nu} A_\alpha \,, \tag{2}$$

$$\begin{aligned} \nabla_\mu A^{\alpha_1..\alpha_a}{}_{\beta_1..\beta_b} &= \partial_\mu A^{\alpha_1..\alpha_a}{}_{\beta_1..\beta_b} + \sum_{c=1}^{a} \Gamma^{\alpha_c}{}_{\mu\gamma} A^{\alpha_1..\alpha_{c-1}\gamma\alpha_{c+1}..\alpha_a}{}_{\beta_1..\beta_b} \\ &\quad - \sum_{c=1}^{b} \Gamma^\gamma{}_{\mu\beta_c} A^{\alpha_1..\alpha_a}{}_{\beta_1..\beta_{c-1}\gamma\beta_{c+1}..\beta_b} \,, \end{aligned} \tag{3}$$

$$\begin{aligned} \nabla_\mu A_{\alpha\beta} &= \partial_\mu A_{\alpha\beta} - \Gamma^\gamma{}_{\mu\alpha} A_{\gamma\beta} - \Gamma^\gamma{}_{\mu\beta} A_{\alpha\gamma} \,, \\ \nabla_\mu A^{\alpha\beta} &= \partial_\mu A^{\alpha\beta} + \Gamma^\alpha{}_{\mu\gamma} A^{\gamma\beta} + \Gamma^\beta{}_{\mu\gamma} A^{\alpha\gamma} \,, \\ \nabla_\mu A^\alpha{}_\beta &= \partial_\mu A^\alpha{}_\beta + \Gamma^\alpha{}_{\mu\gamma} A^\gamma{}_\beta - \Gamma^\gamma{}_{\mu\beta} A^\alpha{}_\gamma \,. \end{aligned} \tag{4}$$

Transformation law for an affine connection :

$$\Gamma'^\rho{}_{\mu\nu} = J'^\rho{}_\gamma J^\alpha{}_\mu J^\beta{}_\nu \Gamma^\gamma{}_{\alpha\beta} + J'^\rho{}_\gamma J^\alpha{}_\mu \partial_\alpha J^\gamma{}_\nu \,. \tag{5}$$

Torsion : $T^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{[\alpha\beta]}$.

Equivalent equations for an affine geodesic :

$$\frac{dT^\mu}{du} + \Gamma^\mu{}_{\alpha\beta} T^\alpha T^\beta = \lambda T^\mu \,, \quad T^\nu \nabla_\nu T^\mu = \lambda T^\mu \,, \tag{6}$$

$$\frac{d^2 x^\mu}{du^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{du}\frac{dx^\beta}{du} = \lambda \frac{dx^\mu}{du} \,. \tag{7}$$

The geodesic is affinely parameterized if $\lambda = 0$.

Parallel Transport :

A vector A^μ parallel transported along a curve is the solution of

$$T^\nu \nabla_\nu A^\mu = 0\,, \quad \frac{dA^\mu}{du} + \Gamma^\mu{}_{\rho\sigma} T^\rho A^\sigma = 0\,. \tag{8}$$

Metric geodesics :

$$S = \int d\tau \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right)^{1/2} d\tau\,, \tag{9}$$

$$\delta S = 0 \quad \to \quad \frac{d^2 x^\mu}{du^2} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0\,, \tag{10}$$

$$\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} = \frac{1}{2} g^{\mu\nu} \left(-\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} \right)\,. \tag{11}$$

Metric connection :

If $\nabla_\alpha g_{\mu\nu} = 0$ and $T^\mu{}_{\alpha\beta} = 0$ then

$$\Gamma^\mu{}_{\alpha\beta} = \left\{ \begin{matrix} \nu \\ \alpha\beta \end{matrix} \right\}\,. \tag{12}$$

Locally inertial coordinates :

By making the coordinate transformation $x'^\nu = x^\mu + \frac{1}{2}\Gamma^\mu{}_{\alpha\beta} x^\alpha x^\beta$ one can set the transformed connection to zero at $x^\mu = 0$.

Killings equation :

An infinitesimal isometry satisfies

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0\,. \tag{13}$$

$T^\mu \xi_\mu$ is conserved along an affinely parameterized geodesic, that is

$$T^\nu \nabla_\nu \left(T^\mu \xi_\mu \right) = 0\,. \tag{14}$$