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MANIFOLDS, METRIC AND TENSORS IN A NUTSHELL

Manifold : a set of 'pieces' which 'looklike' open subsets of R^n such that they can be 'sewn' together smoothly.

Hypersurface : a surface of dimension $(n-1)$ in an n-manifold.

Coordinate transformations : $x^\mu \rightarrow x'^\mu$

$$dx'^\mu = J'^\mu{}_\nu \, dx^\nu \,, \quad dx^\mu = J^\mu{}_\nu \, dx'^\nu \,, \quad J'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \,, \quad J^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \,, \quad J^\mu{}_\nu \, J'^\nu{}_\rho = \delta^\mu{}_\rho \qquad (1)$$

Metrics : A way of measuring distances on a manifold

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \quad g'_{\alpha\beta} = J^\mu{}_\alpha J^\nu{}_\beta \, g_{\mu\nu} \,, \quad g_{\mu\nu} \, g^{\nu\rho} = \delta_\mu{}^\rho \qquad (2)$$

Tangent vectors : For a curve $x^\mu(u)$

$$T^\mu = \frac{dx^\mu}{du} \,, \quad T'^\mu = J'^\mu{}_\nu \, T^\nu \qquad (3)$$

Vectors and co-vectors : A vector is an array, defined at a point in a manifold which transforms like a tangent vector. A co-vector transforms under the inverse.

$$A'^\mu = J'^\mu{}_\nu \, A^\nu \,, \quad A^\mu = J^\mu{}_\nu \, A'^\nu \,, \quad A'_\mu = J^\alpha{}_\mu \, A_\alpha \,, \quad A_\mu = J'^\nu{}_\mu \, A'_\nu \qquad (4)$$

$$A_\mu = g_{\mu\nu} A^\nu \,, \quad A^\mu = g^{\mu\nu} A_\nu \qquad (5)$$

Tensors : An $\begin{pmatrix} a \\ b \end{pmatrix}$ tensor transforms like

$$A'^{\mu_1..\mu_a}{}_{\nu_1..\nu_b} = J'^{\mu_1}{}_{\alpha_1} .. J'^{\mu_a}{}_{\alpha_a} \, J^{\beta_1}{}_{\nu_1} .. J^{\beta_b}{}_{\nu_b} \, A^{\alpha_1..\alpha_a}{}_{\beta_1..\beta_b} \qquad (6)$$

Inner product :

$$d(S,T) = S_\mu T^\mu = g_{\mu\nu} T^\mu S^\nu \,, \quad ||T|| = d(T,T)^{1/2} \,, \quad \cos\theta = \frac{d(S,T)}{||T|| \, ||S||} \qquad (7)$$

Conformal metrics :

$$G_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad (8)$$