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METRIC GEODESICS

In relativity we extremize the action

$$S = \int d\tau \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right)^{1/2} . \tag{1}$$

Hence, we want to compute the E-L equations with

$$L = \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right)^{1/2} . \tag{2}$$

The derivatives are

$$\frac{\partial L}{\partial \left(\frac{\partial x^\mu}{d\tau} \right)} = \frac{g_{\alpha\beta} \delta^\alpha{}_\mu \frac{dx^\beta}{d\tau} + g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \delta^\beta{}_\mu}{L} , \tag{3}$$

and

$$\frac{\partial L}{\partial x^\mu} = \frac{\frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \partial_\mu g_{\alpha\beta}}{L} . \tag{4}$$

Therefore, the E-L equation is

$$\frac{d}{d\tau} \left(\frac{g_{\mu\beta} \frac{dx^\beta}{d\tau} + g_{\alpha\mu} \frac{dx^\alpha}{d\tau}}{L} \right) - \frac{\frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \partial_\mu g_{\alpha\beta}}{L} = 0 . \tag{5}$$

One can the use the fact that

$$\frac{d}{d\tau} g_{\mu\beta} = \frac{dx^\gamma}{d\tau} \partial_\gamma g_{\mu\beta} , \tag{6}$$

to deduce that

$$\frac{d^2 x^\nu}{d\tau^2} + \left\{ \begin{array}{c} \nu \\ \alpha\,\beta \end{array} \right\} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 , \tag{7}$$

where

$$\left\{ \begin{array}{c} \nu \\ \alpha\,\beta \end{array} \right\} = \frac{1}{2} g^{\nu\mu} \left(-\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} \right) . \tag{8}$$

METRIC CONNECTION

The metric connection is defined by $\nabla_\alpha g_{\mu\nu} = 0$. From this we can deduce that

$$\begin{aligned} \nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\gamma{}_{\alpha\mu} g_{\gamma\nu} - \Gamma^\gamma{}_{\alpha\nu} g_{\mu\gamma} = 0 \\ \nabla_\nu g_{\alpha\mu} &= \partial_\nu g_{\alpha\mu} - \Gamma^\gamma{}_{\nu\alpha} g_{\gamma\mu} - \Gamma^\gamma{}_{\nu\mu} g_{\alpha\gamma} = 0 \\ \nabla_\mu g_{\nu\alpha} &= \partial_\mu g_{\nu\alpha} - \Gamma^\gamma{}_{\mu\nu} g_{\gamma\alpha} - \Gamma^\gamma{}_{\mu\alpha} g_{\nu\gamma} = 0 . \end{aligned} \tag{9}$$

Subtract the first from the sum of the other two, to give

$$\left(\Gamma^{\gamma}{}_{\nu\mu} + \Gamma^{\gamma}{}_{\mu\nu}\right) g_{\alpha\gamma} + \left(\Gamma^{\gamma}{}_{\mu\alpha} - \Gamma^{\gamma}{}_{\alpha\mu}\right) g_{\gamma\nu} + \left(\Gamma^{\gamma}{}_{\alpha\nu} - \Gamma^{\gamma}{}_{\nu\alpha}\right) g_{\mu\gamma} = -\partial_{\alpha} g_{\mu\nu} + \partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\nu\alpha} \ , \quad (10)$$

from which one can deduce that

$$\Gamma^{\rho}{}_{\nu\mu} + \Gamma^{\rho}{}_{\mu\nu} = g^{\alpha\rho} \left(-\partial_{\alpha} g_{\mu\nu} + \partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\nu\alpha}\right) + 2 \left(T_{\nu}{}^{\rho}{}_{\mu} + T_{\mu}{}^{\rho}{}_{\nu}\right) \ . \quad (11)$$

If we assume that the connection is torsion free, that is, $T^{\mu}{}_{\alpha\beta} = 0$, then

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} \left(-\partial_{\alpha} g_{\mu\nu} + \partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\mu\alpha}\right) = \left\{ \begin{array}{c} \rho \\ \mu\,\nu \end{array} \right\} \ , \quad (12)$$

that is, an affine connection which is torsion free and $\nabla_{\alpha} g_{\mu\nu} = 0$, this the metric connection.