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3.6 COMPUTING CHRISTOFFEL SYMBOLS

Direct computation of the Christoffel symbols, and hence the geodesics requires $\frac{1}{2}n^2(n+1)$ evaluations which can be become very cumbersome! (n is the dimension of the spacetime)

We have shown that the Geodesic equations can be derived by extremizing the functional

$$L = \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right)^{\frac{1}{2}} . \tag{1}$$

The square root makes things a little computationally difficult. However, since L is constant, any function of L will yield the same Euler-Lagrange equation. In particular we can use L^2:

$$\frac{\partial}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\left\{L^2\right\} = 2L\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)} \quad \text{and} \quad \frac{\partial}{\partial x^\mu}\left\{L^2\right\} = 2L\frac{\partial L}{\partial x^\mu} \tag{2}$$

$$\rightarrow \frac{d}{d\tau}\left[\frac{\partial}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\left\{L^2\right\}\right] - \frac{\partial}{\partial x^\mu}\left\{L^2\right\}$$

$$= 2L\left[\frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\right) - \frac{\partial L}{\partial x^\mu}\right] = 0 \tag{3}$$

ie if L satisfies the E-L equation then so does L^2.

Hence, we can compute the geodesic equations and read off the Christoffel symbols. If one requires the geodesics equations non-parameterically then one should use L directly and use one of the coordinates as the parameter along the curve.

Example :

$$ds^2 = \frac{1}{t^2}\left(dt^2 - dx^2\right) \tag{4}$$

$$\rightarrow \quad g = \begin{pmatrix} 1/t^2 & 0 \\ 0 & -1/t^2 \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix} \tag{5}$$

A: Direct evaluation of Christoffel Symbols

The spacetime is 2 dimensional and therefore there are 6 independent components. They are

$$\Gamma^t{}_{tt}\,, \quad \Gamma^t{}_{xt}\,, \quad \Gamma^t{}_{xx}\,, \quad \Gamma^x{}_{tt}\,, \quad \Gamma^x{}_{xt}\,, \quad \Gamma^x{}_{xx}\,. \tag{6}$$

Remember the connection is symmetric and therefore $\Gamma^x{}_{xt} = \Gamma^x{}_{tx}$ and $\Gamma^t{}_{xt} = \Gamma^t{}_{tx}$.

The expression for the Christoffel symbols in terms of the metric is

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(-\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\rho\beta} + \partial_\beta g_{\alpha\rho} \right) . \tag{7}$$

In order to systematically compute the symbols one should first consider the each of the cases $\mu = t$ and $\mu = x$ separately.

For the case of $\mu = t$

$$\begin{aligned} \Gamma^{t}{}_{ij} &= \frac{1}{2} g^{t\rho} \left(-\partial_\rho g_{ij} + \partial_i g_{\rho j} + \partial_j g_{i\rho} \right) \\ &= \frac{1}{2} g^{tt} \left(-\partial_t g_{ij} + \partial_i g_{tj} + \partial_j g_{it} \right) . \end{aligned} \tag{8}$$

Then one considers the cases of $(i,j) = (t,t), (x,x), (x,t)$ separately

$$\Gamma^{t}{}_{tt} = \frac{1}{2} g^{tt} \frac{\partial}{\partial t} g_{tt} = -\frac{1}{t} , \quad \Gamma^{t}{}_{xt} = 0 , \quad \Gamma^{t}{}_{xx} = -\frac{1}{2} g^{tt} \frac{\partial}{\partial t} g_{tt} = -\frac{1}{t} . \tag{9}$$

For the case of $\mu = x$

$$\begin{aligned} \Gamma^{x}{}_{ij} &= \frac{1}{2} g^{x\rho} \left(-\partial_\rho g_{ij} + \partial_i g_{\rho j} + \partial_j g_{i\rho} \right) \\ &= \frac{1}{2} g^{xx} \left(-\partial_x g_{ij} + \partial_i g_{xj} + \partial_j g_{ix} \right) , \end{aligned} \tag{10}$$

and

$$\Gamma^{x}{}_{tt} = 0 , \quad \Gamma^{x}{}_{xt} = \frac{1}{2} g^{xx} \frac{\partial}{\partial t} g_{xx} = -\frac{1}{t} , \quad \Gamma^{x}{}_{xx} = 0 . \tag{11}$$

If the geodesics are parameterized by u and $\dot{t} = \frac{dt}{du}$ and $\dot{x} = \frac{dx}{du}$ then

$$\begin{aligned} &\ddot{t} + \Gamma^{t}{}_{tt} \dot{t}^2 + 2\Gamma^{t}{}_{xt} \dot{t}\dot{x} + \Gamma^{t}{}_{xx} \dot{x}^2 = 0 , \\ &\ddot{x} + \Gamma^{x}{}_{tt} \dot{t}^2 + 2\Gamma^{x}{}_{xt} \dot{t}\dot{x} + \Gamma^{x}{}_{xx} \dot{x}^2 = 0 , \end{aligned} \tag{12}$$

and therefore

$$\ddot{t} - \frac{1}{t} \left(\dot{t}^2 + \dot{x}^2 \right) = 0 , \quad \ddot{x} - \frac{2}{t} \dot{t}\dot{x} = 0 . \tag{13}$$

B: Using $L_{\text{eff}} = L^2$

$$L_{\text{eff}} = L^2 = \frac{\dot{t}^2 - \dot{x}^2}{t^2} \tag{14}$$

$$\frac{\partial L_{\text{eff}}}{\partial \dot{t}} = \frac{2\dot{t}}{t^2} , \quad \frac{\partial L_{\text{eff}}}{\partial \dot{x}} = -\frac{2\dot{x}}{t^2} , \quad \frac{\partial L_{\text{eff}}}{\partial t} = -\frac{2}{t^3} \left(\dot{t}^2 - \dot{x}^2 \right) . \tag{15}$$

$$\ddot{x} - \frac{2}{t} \dot{x}\dot{t} = 0 , \quad \ddot{t} - \frac{\dot{t}^2}{t} - \frac{\dot{x}^2}{t} = 0 . \tag{16}$$

Using (**??**) and the equations above one can read off

$$\Gamma^x{}_{xt} = -\frac{1}{t}\,, \quad \Gamma^t{}_{xx} = -\frac{1}{t}\,, \quad \Gamma^t{}_{tt} = -\frac{1}{t}\,. \tag{17}$$

C: Direct computation of the geodesics

$$L^2 = \frac{1}{t^2}\left[1 - \left(\frac{dx}{dt}\right)^2\right] \quad \text{or} \quad \frac{1}{t^2}\left[\left(\frac{dt}{dx}\right)^2 - 1\right]. \tag{18}$$

We will use

$$L = \frac{1}{t}\left(1 - x'^2\right)^{1/2} \quad \text{where} \quad x' = \frac{dx}{dt}\,. \tag{19}$$

From which one can deduce

$$\frac{\partial L}{\partial x'} = -\frac{x'}{t(1-x'^2)^{1/2}} \quad \rightarrow \quad \frac{x'}{t(1-x'^2)^{1/2}} = A = \text{constant}\,. \tag{20}$$

$$\frac{dx}{dt} = \pm\frac{At}{(1+A^2t^2)^{1/2}} \quad \rightarrow \quad (Ax+B)^2 = 1 + A^2t^2\,, \tag{21}$$

The geodesics are hyperbolae.