GRAVITATION PC4771 2003 : Dr R.A. Battye

EXTRA TENSOR AND LAGRANGIAN DYNAMICS QUESTIONS

1. Summation convention, matrices and vectors

Prove that :

(i) The trace of the product of a symmetric and an antisymmetric matrix is zero.

(ii) If A is an antisymmetric matrix then $\mathbf{x}^T A\mathbf{x} = 0$ for any vector $\mathbf{x}$.

2. Projection tensors

Show that if $R_{ij}s_j$ is a vector for any vector s_j then R_{ij} is a tensor. Show that if $R_{ij}s_j = 0$ for every vector s_j then $R_{ij} = 0$.

Given a non-zero vector v_i, the projection tensor P_{ij} associated with it has the following properties

(i) $P_{ij}v_j = 0$
(ii) if a vector u_i is orthogonal to v_i then $P_{ij}u_j = u_i$.

Show that

$$P_{ij} = \delta_{ij} - \frac{v_i v_j}{v_k v_k}\,, \tag{1}$$

satisfies the conditions (i) and (ii).

Furthermore, show that if T_{ij} is another tensor which satisfies (i) and (ii) then

$$(T_{ij} - P_{ij})w_j = 0\,, \tag{2}$$

for any vector w_i. Hence show that the projection tensor is unique and that the associated opertaor satisfies $P^2 = P$.

3. Isotropic tensors

In the theory of piezo-electricity, the electric polarization density P_i is induced by a mechanical stress described by a symmetric 2nd rank tensor S_{ij} via the linear relation

$$P_i = d_{ijk}S_{jk}\,, \tag{3}$$

where d_{ijk} is a 3rd rank tensor only dependent on the material. Show that P_i can be non-zero only in anisotropic materials.

4. Symmetric and antisymmetric tensors

Show that if T_{ij} is a 2nd rank tensor, antisymmetric in one frame then it is antisymmetric in all frames and that it can be written in terms of a vector w_i according to

$$T_{ij} = \epsilon_{ijk}w_k\,. \tag{4}$$

Furthermore, show that it can be written in terms of two vectors u_i and v_i such that

$$T_{ij} = u_i v_j - v_i u_j \,. \tag{5}$$

Is the choice of u_i and v_i unique?

5. Eigenvalues and angles between vectors

(i) Show that the eigenvalues of a 2nd rank tensor are unaffected by a coordinate rotation. How are the eigenvectors affected by the rotation?
(ii) If u_i is a vector and σ_{ij} is a 2nd rank symmetric tensor then show that $v_i = \sigma_{ij} u_j$ is a vector and that the angle between u_i and v_j is unaffected by a coordinate rotation.

6. Principal axes

The conductivity tensor for an homogeneous material in a Cartesian coordinate system has components

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \,. \tag{6}$$

Find the directions in space along which

(i) no current can travel

(ii) the conductivity is greatest

Calculate the conductivity in the latter direction or directions.

7. Fermat's Principle

Fermat's principle of optics states that a light ray will follow the path, $y(x)$, for which

$$S = \int_{(x_1,y_1)}^{(x_2,y_2)} n(x,y) ds \,, \tag{7}$$

is minimized where $n(x,y)$ is the refractive index. If $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (1, 1)$ find the path of light if (a) $n = e^y$,

(b) $n = a(y - y_0)$ where a and y_0 are positive constants.

8. Moving Particle in Polar coordinates

Consider a particle of mass m moving in the plane under the influence of no potential. Working in polar coordinates (ρ, θ) , the kinetic energy is

$$T = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 \right) \,. \tag{8}$$

Deduce a Lagrangian and compute the equations of motion and deduce the constants of motion.

9. Relativistic Particle Motion

Show that the Lagrangian

$$L = m_0 \left(1 - \sqrt{1 - v^2}\right) - V(x)\,, \tag{9}$$

leads to a relativistic form of Newton's 2nd Law of notion

$$\frac{d}{dt}\left(\frac{m_o v}{\sqrt{1 - v^2}}\right) = F \tag{10}$$

where $F = -dV/dx$.

10. Simple example

Compute the function $y(x)$ which minimizes the functional

$$\int_0^1 dx\, y^2 y'^2\,, \tag{11}$$

between the points $(0, 1)$ and $(1, 2)$.