**GRAVITATION PC4771 2003 : Dr R.A. Battye**

**EXAMPLES SHEET 3 : GEODESICS AND CURVATURE**

**1. Reparameterization of geodesics**

A curve $x^\mu(\lambda)$ is a geodesic if

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = f(\lambda)\frac{dx^\mu}{d\lambda}\,. \tag{1}$$

Show that any geodesic can be affinely parameterized by a suitable change of parameter $\rho(\lambda)$. If $\lambda$ is an affine parameter, show that the most general affine parameter is $\rho = A\lambda + B$, where $A \neq 0$ and $B$ are constants.

**2. Compute grad, div and the Laplacian for spherical polars**

The Cartesian coordinates can be converted to spherical polar coordinates via

$$x = r\sin\theta\cos\phi\,,\quad y = r\sin\theta\sin\phi\,,\quad z = r\cos\theta\,.$$

The gradient of a scalar field $s$ is the defined as $g^{\mu\nu}\partial_\nu s$, the divergence of a vector field $V^\mu$ is $\nabla_\mu V^\mu$ and the Laplacian of the scalar field is given by $g^{\mu\nu}\nabla_\mu\nabla_\nu s$. Compute them in spherical polar coordinates.

**3. Nordstroms theory of gravity**

If $\Gamma^\mu{}_{\alpha\beta}$ is the metric connection associated with $g_{\mu\nu}$, show that associated with $\hat{g}_{\mu\nu} = \Omega^2\, g_{\mu\nu}$ is given by

$$\hat{\Gamma^\mu}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} + \frac{1}{\Omega}\left[\delta^\mu{}_\alpha\partial_\beta\Omega + \delta^\mu{}_\beta\partial_\alpha\Omega - g^{\mu\delta}g_{\alpha\beta}\partial_\delta\Omega\right]\,. \tag{2}$$

Before Einstein postulated the Einstein equations, Nordstrom suggested a theory of gravity defined by $\hat{g}_{\mu\nu} = e^{2\varphi}\eta_{\mu\nu}$ where $\varphi$ is the gravitational potential and $\eta_{\mu\nu}$ is the Minkowski metric. Compute the geodesic equation for Nordstroms theory of gravity. Using the fact that geodesics can be affinely parameterized (see question 1) show that null geodesics in Nordstrom's theory are also geodesics in Minkowski space. What can be said about freely falling massive particles?

**4. Pseudo forces**

Consider a particle with coordinates $(t, x, y, z)$ in a frame $S$. Compute the Jacobian matrix for the coordinate transformation

$$(t', x', y', z') = \left(t, x - \frac{1}{2}gt^2, y, z\right)\,.$$

Compute the metric, Christoffel symbols and geodesic equations in the new coordinate system and identify the nature of the pseudo forces.

**5. 2D de-Sitter spacetime**

Consider the line elemenet

$$ds^2 = du^2 - \cosh^2 u \, d\phi^2 \,, \tag{3}$$

where $-\infty < u < \infty$ and $0 \leq \phi < 2\pi$. Compute the geodesic equations parameterized by $\lambda$ and hence deduce the Christoffel symbols for the metric. Show that there are two first integrals

$$\cosh^2 u \, \dot{\phi} = K \,, \quad \cosh^2 u \, \dot{u}^2 = K^2 + L \cosh^2 u \,, \tag{4}$$

where $K$ and $L$ are constants.

If $v = \tanh u$ and $K \neq 0$ the show that

$$\left(\frac{dv}{d\phi}\right)^2 = M^2 - v^2 \,, \tag{5}$$

where $M$ is a constant which is $> 1$ on timelike geodesics, $= 1$ on null geodesics and $< 1$ on spacelike geodesics. Hence, compute the $u(\phi)$.

**6. Classification of geodesics**

Show that

$$L^2 = \left(\frac{ds}{d\tau}\right)^2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \,, \tag{6}$$

is conserved along an affinely parameterized geodesic if the connection is the metric connection. What does this imply about for the classification of geodesics as timelike, null and spacelike?

**7. Commutator for 2nd rank tensor**

If $U^\alpha$ and $V^\beta$ are vectors evaluate

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \, U^\alpha V^\beta \,, \tag{7}$$

and hence show that

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \, T^{\alpha\beta} = R^\alpha{}_{\gamma\nu\nu} T^{\gamma\beta} + R^\beta{}_{\gamma\nu\mu} T^{\alpha\gamma} \,, \tag{8}$$

for an arbitrary $\binom{2}{0}$ tensor $T^{\alpha\beta}$. If $S^{\mu\nu} = -S^{\nu\mu}$ then show that $\nabla_\mu \nabla_\nu S^{\mu\nu}$ and deduce that

$$\nabla_\mu \nabla_\nu T^{\mu\nu} = \nabla_\nu \nabla_\mu T^{\mu\nu} \,, \tag{9}$$

for any $\binom{2}{0}$ tensor $T^{\mu\nu}$

**8. Killing's equation**

Consider an isometry $\xi_\mu$. Using the symmetry relation $R^\alpha{}_{[\beta\gamma\delta]} = 0$ to show that

$$\nabla_\gamma \nabla_\delta \xi_\beta = R^\alpha{}_{\gamma\delta\beta} \xi_\alpha \,. \tag{10}$$

Minkowski spacetime is flat. Use this fact to solve the Killing equation and identify the independent Killing vectors. How many are there?

**9. Newtonian connection**

Consider the equations of motion for Newtonian gravity. Show that a freely falling particle moves along a curve $x^\mu = x^\mu(\lambda) = (t, x^i)$ where

$$\frac{d^2 t}{d\lambda^2} = 0\,, \quad \frac{d^2 x^i}{d\lambda^2} + \partial^i \varphi \left(\frac{dt}{d\lambda}\right)^2 = 0\,, \tag{11}$$

for a gravitational potential $\varphi(x^i)$ and deduce a Newtonian connection. Show that

$$R_{i0j0} = -R_{0i0j} = \partial_i \partial_j \varphi, \,, \tag{12}$$

and hence that the Riemman tensor does not respect the standard symmetry relation. Can the Newtonian connection be written in terms of a metric?