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EXAMPLES SHEET 2 : MANIFOLDS, METRICS AND TENSORS

1. Maps between sets

Decide whether the following maps are injective, surjective or bijective, and identify the domain and image.

$$\text{(i) } f(x) = e^x,\ x \in R \quad \text{(ii) } f(x) = x^3 - x,\ x \in R \quad \text{(iii) } f(x) = x^3,\ x \in R \tag{1}$$

$$\text{(iv) } f(x) = x^2,\ x \in R \quad \text{(v) } f(x) = \ln(x),\ x \in R,\ x > 0 \tag{2}$$

2. Hyperbolic coordinates

By identifying the specific coordinate transformation $x(u,v)$ and $t(u,v)$, or otherwise, show that the 2D metric given by the line element

$$ds^2 = dv^2 - v^2 du^2\,, \tag{3}$$

is in fact equivalent to that of 2D Minkowski space.

3. Mercators projection

The most famous flat map of the earth is a projection of the sphere onto R^2 according to

$$x = \phi\,, \quad y = \log\left(\cot\frac{1}{2}\theta\right)\,. \tag{4}$$

From standard line element for the sphere

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,, \tag{5}$$

compute the metric for the projection and the geodesics. Comment on the implications for sailors!

4. Degenerate metrics

Consider the line element

$$ds^2 = dx^2 + dy^2 + dz^2 - (\alpha dx + \beta dy + \gamma dz)^2\,. \tag{6}$$

Show that the space time (x,y,z) has a degenerate direction if $\alpha^2 + \beta^2 + \gamma^2 = 1$, and can be thought of as effetively two dimensional. NB a degenerate metric has zero determinant.

If $\alpha = 3/13$, $\beta = 4/13$ and $\gamma = 12/13$, transform the metric into canonical form. Is the metric Euclidean or Lorentzian?

5. Coordinate transformation of vectors

Consider a vector A^μ in a 2D cartesian coordinate system (x, y). Derive the transformation matrix to convert to 2D polar coordinates (r, θ) and convert

$$\text{(i) } A^\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{(ii) } A^\mu = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{7}$$

into the form

$$A^\mu = \begin{pmatrix} A_r \\ A_\theta \end{pmatrix}. \tag{8}$$

6. Tensor transformation and contraction

If $A^{\mu\nu}$ is a tensor of type $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $B_{\mu\nu}$ is a tensor of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ then show that

$$C^{\mu\nu}{}_{\rho\delta} = A^{\mu\nu} B_{\rho\delta}\,, \tag{9}$$

is a tensor of type $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. What type of tensors are

$$\text{(i) } g_{\mu\nu} C^{\mu\nu}{}_{\rho\delta}, \quad \text{(ii) } g^{\rho\delta} C^{\mu\nu}{}_{\rho\delta}, \quad \text{(iii) } g_{\mu\nu} g^{\rho\delta} C^{\mu\nu}{}_{\rho\delta}. \tag{10}$$

7. Tensor algebra

Simplify in an n-dimensional curved space :

$$\text{(i) } \delta^\nu{}_\mu\, \delta^\rho{}_\nu\, \delta^\gamma{}_\rho, \quad \text{(ii) } \delta^\mu{}_\mu, \quad \text{(iii) } g^{\mu\nu} g_{\alpha\beta}\, T^\alpha{}_{\nu\gamma}, \quad \text{(iv) } g^{\mu\nu}\, T_{\mu\nu}. \tag{11}$$

8. Kronecker delta

Show that $\delta^\mu{}_\nu$ is an isotropic tensor, ie. it has the same numerical value in all coordinate systems.

9. Symmetrization and anti-symmetrization

(i) Expand :

$$T_{\mu[\nu\rho]} \quad \text{and} \quad T_{(\mu\nu)\rho}. \tag{12}$$

(ii) Show that :

$$T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}. \tag{13}$$

10. Covariant derivatives as tensors

Without using the tensor transformation law directly, show the following are tensors if the connection is torsion free.

$$\text{(i) } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{14}$$

$$\text{(ii) } [U, V]^\mu = U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu U^\mu\,. \tag{15}$$

11. Covariant derivative of a mixed tensor

If $A^\mu = B^\mu{}_\nu C^\nu$, A^μ, C^μ are vectors and $B^\mu{}_\nu$ is a $\begin{pmatrix}1\\1\end{pmatrix}$ tensor then derive the formula for the covariant derivative for $B^\mu{}_\nu$ from that for the vectors

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\alpha} A^\alpha \,. \tag{16}$$

Use this formula to compute

$$\nabla_\mu \, \delta^\alpha{}_\beta \,. \tag{17}$$

What is the implication of this result for the relation between covariant differentiation and contraction?