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EXAMPLES SHEET 1 : PRELIMINARY AND REVISION MATERIAL

1. Summation convention

Write the following matrix operations in summation notation

$$(i)\,\mathrm{Tr}(A)\,,\quad (ii)\,AB\,,\quad (iii) AB^T\,,\quad (iv)\,ABC\,,\quad (v)\mathrm{Tr}(AB)\,. \tag{1}$$

2. Tensor detection theorem

A_{ijkr} is an array of numbers which is symmetric under interchange of the first two suffices. If U_{kr} is a tensor of rank 2) defined by

$$U_{kr} = V_i V_j A_{ijkr}\,, \tag{2}$$

for any arbitrary vector V_i, prove that A_{ijkr} is tensor of rank 4.

3. Isotropic tensors

In a linearly elastic medium the strain tensor e_{ij} and the stress tensor p_{kl} are related linearly, that is,

$$e_{ij} = A_{ijkl} p_{kl}\,, \tag{3}$$

where A_{ijkl} is a tensor of rank 4. Both the strain and stress are symmetric tensors. If the medium is isotropic, then show that the relation between strain and stress can be characterized in terms of two coefficients E and σ via

$$E e_{ik} = (1+\sigma) p_{ik} - \sigma p_{mm} \delta_{ik}\,. \tag{4}$$

4. Calculus of variations I

Find the path y(x) which lies in the strip $\{(x,y) : 0 \leq x \leq 1, y > 0\}$ which satisfies boundary conditions $y(0) = 1$, $y(1) = 9$ and minimizes the integral

$$I = \int_0^1 \frac{dx}{y} \left(\frac{dy}{dx}\right)^2\,. \tag{5}$$

5. Calculus of variations II

A tunnel is to be drilled between the points $(-1,0)$ and $(1,0)$ in the upper half plane. The cost is inversely proportional to y, the distance above the straight line connecting the two points. Formulate a functional for the cost of the project and find the path $y(x)$ which minimizes the cost.

6. Newtonian Mechanics from calculus of variations

(a) Compute the Euler-Lagrange equations for

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\,, \tag{6}$$

where V is the potential energy and compare it to Newton's equations of motion for gravity.

(b) Consider the motion of a particle of mass m in spherical polar coordinates (r, θ, ϕ) where $\mathbf{x} = r\hat{\mathbf{r}}$ and

$$\hat{\mathbf{r}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}\,, \tag{7}$$

under the influence of a potential $V(r)$. Establish a Lagrangian formulation for the dynamics, compute the Euler-Lagrange equations and the relevant first integrals. Show that that the motion along the equator $\theta = \pi/2$ a stable solution of the E-L equations. What is the physical significance of the first integrals? What are conditions for a circular stable orbit if

$$V(r) = -\frac{C}{r^3}\,, \tag{8}$$

where $C > 0$.

7. Tidal gravitational fields

Two Galilean reference frames (x, t) and (x', t') are related by

$$t' = t + a\,, \quad \mathbf{x}' = \mathbf{x} + \mathbf{b}(t)\,, \tag{9}$$

where a is a constant and $\mathbf{b}(t)$ is a vector valued function of t. In the first frame a particle moves along a trajectory $\mathbf{x}(t)$. Write down expressions for the velocity and acceleration as measured in each frame.

The particle moves solely under the influence of gravity, i.e $\ddot{\mathbf{x}} = -\nabla\phi$, where $\phi(\mathbf{x}, t)$ is the Newtonian potential in the first frame. What is the Newtonian potential in the other frame?

Hence, show that it is always possible to set the potential to zero at point by a change of coordinates, but that the tidal gravitational fields are frame independent.

8. Lorentz transformation I

(a) Rewrite the wave equation for a scalar field

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}\,, \tag{10}$$

in terms of the coordinate $x_\pm = x \pm t$ and use it deduce the general solution in terms of two arbitrary functions.

(b) Show that under the Lorentz transformation the coordinates transform according to

$$x_{\pm} \quad \to \quad x'_{\pm} = \sqrt{\frac{1 \mp v}{1 \pm v}}\, x_{\pm} \,, \tag{11}$$

where v is the velocity associated with the transformation. Hence, show that if $\phi(x_+, x_-)$ is a solution of the wave equation then so is $\phi(x'_+, x'_-)$.

(c) By considering a right-moving plane wave solution

$$\phi = A e^{ik(x-t)} \,. \tag{12}$$

deduce the Doppler shift formula.

9. Lorentz transformation II

In an inertial frame S a photon with energy E moves in the xy-plane at an angle, θ, relative to the x-axis. Show that in another frame S' moving with a velocity v in the x-direction in S, the energy and the angle are given by

$$E' = \gamma E(1 - v\cos\theta) \quad \text{and} \quad \cos\theta' = \frac{\cos\theta - v}{1 - v\cos\theta} \,, \tag{13}$$

where $\gamma = (1 - v^2)^{-1/2}$. Invert these relations to give E and $\cos\theta$ in terms of E' and $\cos\theta'$.

(a) Show that a photon moving in the x-direction experiences a frequency change which you should determine.

(b) Next consider a source of photons emitted in the forward direction ($\cos\theta' > 0$) which is at rest in S'. Show that if $v \approx 1$ these photons will appear to be concentrated in a narrow cone about $\theta = 0$ in S.

10. Relativistic particle mechanics I

The unstable π^- particle of rest mass m_π can decay into an electron of rest mass m_e and neutrino (ν) of zero rest mass via the process

$$\pi^- \to e^- + \nu \,. \tag{14}$$

Given that the π^- particle is at rest initially, obtain a formula for the speed v of the electron as a function of the ratio m_e/m_π, and the angle between the motion of the electron and the neutrino.

11. Relativistic particle mechanics II

In a laboratory frame a particle of rest mass m_1 has energy E_1 and a second particle of rest mass m_2 is at rest. Show that the combined energy in the centre of momentum frame is

$$\sqrt{m_1^2 + m_2^2 + 2E_1 M_2} \,. \tag{15}$$

Hence show that in a collision of a proton of mass m and energy E with an identical one at rest, it is energetically possible to create a proton-antiproton pair if $E \geq 7m$.

12. Gravitational redshift

Consider a particle of rest mass m intially at a point A which drops a distance h to the point B. When the particle gets to B let all its energy be converted into a photon by some annihiltion process which travels back toward A. If the photon does not interact with the gravitational field, what mass particle could it turn back into at A? What is the implication of this simple thought experiment for the conservation of energy? How is this paradox resolved?