**GRAVITATION PC4771 2003 : Dr R.A. Battye**

**DERIVING THE EXPRESSION FOR THE RIEMANN TENSOR**

Consider first,

$$\nabla_\mu B_\nu{}^\rho - \nabla_\nu B_\mu{}^\rho = \partial_\mu B_\nu{}^\rho - \partial_\nu B_\mu{}^\rho + \Gamma^\rho{}_{\mu\gamma} B_\nu{}^\gamma - \Gamma^\rho{}_{\nu\gamma} B_\mu{}^\gamma \,. \tag{1}$$

Now choose,

$$B_\mu{}^\nu = \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\alpha} A^\alpha \,, \tag{2}$$

and substituting into (**??**)

$$\begin{aligned} \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A^\rho &= \partial_\mu \left[\partial_\nu A^\rho + \Gamma^\rho{}_{\nu\alpha} A^\alpha\right] - \partial_\nu \left[\partial_\mu A^\rho + \Gamma^\rho{}_{\mu\alpha} A^\alpha\right] \\ &+ \Gamma^\rho{}_{\nu\gamma} \left[\partial_\mu A^\gamma + \Gamma^\gamma{}_{\nu\alpha} A^\alpha\right] - \Gamma^\rho{}_{\nu\gamma} \left[\partial_\mu A^\gamma + \Gamma^\gamma{}_{\mu\alpha} A^\alpha\right] \,. \end{aligned} \tag{3}$$

Therefore,

$$\begin{aligned} \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A^\rho &= \left(\partial_\mu \Gamma^\rho{}_{\alpha\nu} - \partial_\nu \Gamma^\rho{}_{\alpha\mu} + \Gamma^\rho{}_{\gamma\mu} \Gamma^\gamma{}_{\alpha\nu} - \Gamma^\rho{}_{\gamma\nu} \Gamma^\gamma{}_{\alpha\mu}\right) A^\alpha \\ &= R^\rho{}_{\alpha\mu\nu} A^\alpha \,. \end{aligned} \tag{4}$$

Now consider

$$\nabla_\mu C_{\nu\rho} - \nabla_\nu C_{\mu\rho} = \partial_\mu C_{\nu\rho} - \partial_\nu C_{\mu\rho} + \Gamma^\gamma{}_{\nu\rho} C_{\mu\gamma} - \Gamma^\gamma{}_{\mu\rho} C_{\nu\gamma} \,, \tag{5}$$

with

$$C_{\mu\nu} = \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha{}_{\mu\nu} A_\alpha \,. \tag{6}$$

As with the first example,

$$\begin{aligned} \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A_\rho &= \partial_\mu \left[\partial_\nu A_\rho - \Gamma^\alpha{}_{\nu\rho} A_\alpha\right] - \partial_\nu \left[\partial_\mu A_\rho - \Gamma^\gamma{}_{\nu\rho} A_\alpha\right] \\ &+ \Gamma^\gamma{}_{\nu\rho} \left[\partial_\mu A_\gamma - \Gamma^\alpha{}_{\mu\gamma} A_\alpha\right] - \Gamma^\gamma{}_{\mu\rho} \left[\partial_\nu A_\gamma - \Gamma^\alpha{}_{\nu\gamma} A_\alpha\right] \,, \end{aligned} \tag{7}$$

and therefore

$$\begin{aligned} \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) A_\rho &= \left(\partial_\nu \Gamma^\alpha{}_{\rho\mu} - \partial_\mu \Gamma^\alpha{}_{\rho\nu} + \Gamma^\gamma{}_{\rho\mu} \Gamma^\alpha{}_{\gamma\nu} - \Gamma^\gamma{}_{\rho\nu} \Gamma^\alpha{}_{\gamma\mu}\right) A_\alpha \\ &= R^\alpha{}_{\rho\nu\mu} A_\alpha \,. \end{aligned} \tag{8}$$

The Riemann curvature tensor is defined to be

$$R^\rho{}_{\alpha\mu\nu} = \partial_\mu \Gamma^\rho{}_{\alpha\nu} - \partial_\nu \Gamma^\rho{}_{\alpha\mu} + \Gamma^\rho{}_{\gamma\mu} \Gamma^\gamma{}_{\alpha\nu} - \Gamma^\rho{}_{\gamma\nu} \Gamma^\gamma{}_{\alpha\mu} \,. \tag{9}$$