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POWER SERIES SOLUTIONS

Eliminate h to give

$$\begin{aligned}
\delta_{\rm m}^{\prime} &= \frac{a}{a^{\prime}}\left[k^2 H - \frac{3\omega_{\rm r}\delta_{\rm r}}{2a^2} - \frac{3\omega_{\rm m}\delta_{\rm m}}{2a}\right]\,, \\
\delta_{\rm r}^{\prime} &= \frac{4}{3}\left(\delta_{\rm m}^{\prime} - \theta_{\rm r}\right)\,, \\
\theta_{\rm r}^{\prime} &= \frac{1}{4}k^2\delta_{\rm r}\,, \\
k^2 H^{\prime} &= -\frac{2\omega_{\rm r}\theta_{\rm r}}{a^2}\,. \qquad (1)
\end{aligned}$$

Objective: to compute the first terms in possible regular power series solutions:

$$\begin{aligned}
\delta_{\rm m} &= \sum_{n=0}^{\infty} c_n^{\rm m}\eta^n\,, \\
\delta_{\rm r} &= \sum_{n=0}^{\infty} c_n^{\rm r}\eta^n\,, \\
\theta_{\rm r} &= \sum_{n=0}^{\infty} c_n^{\theta}\eta^n\,, \\
H &= \sum_{n=0}^{\infty} c_n^{\rm H}\eta^n\,. \qquad (2)
\end{aligned}$$

Note that $a \approx \sqrt{\omega_{\rm r}}\eta + \omega_{\rm m}\eta^2/4$ and $a^{\prime} \approx \sqrt{\omega_{\rm r}} + \omega_{\rm m}\eta/2$ since we are only considering only early times.

The last three of the equations of motion in (1) yield :

$$c_1^{\rm r} + 2c_2^{\rm r}\eta + .. = \frac{4}{3}\left(c_1^{\rm m} + 2c_2^{\rm m}\eta - c_0^{\theta} - c_1^{\theta}\eta + ..\right)\,, \qquad (3)$$

$$c_1^{\theta} + 2c_2^{\theta}\eta + 3c_3^{\theta}\eta^2 + .. = \frac{1}{4}k^2\left(c_0^{\rm r} + c_1^{\rm r}\eta + c_2^{\rm r}\eta^2 + ..\right)\,, \qquad (4)$$

$$k^2\left(c_1^{\rm H} + 2c_2^{\rm H}\eta + ..\right) = \frac{2}{\eta^2}\left(1 + \frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\theta} + c_1^{\theta}\eta + c_2^{\theta}\eta^2 + c_3^{\theta}\eta^3 + ..\right)\,. \qquad (5)$$

From these we can deduce that :

$$c_1^{\rm r} = \frac{4}{3}c_1^{\rm m} - c_0^{\theta} \qquad c_2^{\rm r} = \frac{2}{3}(2c_2^{\rm m} - c_1^{\theta})\,, \qquad (6)$$

$$c_1^\theta = \frac{1}{4}k^2 c_0^{\rm r} \qquad c_2^\theta = \frac{1}{8}k^2 c_1^{\rm r} \qquad c_3^\theta = \frac{1}{12}k^2 c_2^{\rm r}\,, \tag{7}$$

$$c_0^\theta = 0 \qquad c_1^\theta = 0 \qquad k^2 c_1^{\rm H} = -2c_2^\theta \qquad k^2 c_2^{\rm H} = -c_3^\theta + \frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}c_2^\theta\,. \tag{8}$$

Hence we can deduce that $c_0^\theta = c_1^\theta = c_0^{\rm r} = 0$.

The first equaiton from (1) implies that :

$$\begin{aligned} c_1^{\rm m} + 2c_2^{\rm m} + .. = \quad & k^2\eta\left(1+\frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}\eta\right)\left(1+\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\rm H} + c_1^{\rm H}\eta + ..\right) \\ - \quad & \frac{3}{2\eta}\left(1+\frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}\eta\right)\left(1+\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\rm r} + c_1^{\rm r}\eta + c_2^{\rm r}\eta^2 + ..\right) \\ - \quad & \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\left(1+\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\rm m} + c_1^{\rm m}\eta + ..\right)\,. \end{aligned} \tag{9}$$

Hence we can deduce that $c_0^{\rm r} = c_0^\theta = c_1^\theta = 0$ and

$$c_1^{\rm m} = -\frac{3}{2}c_1^{\rm r} - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}c_0^{\rm m}\,, \tag{10}$$

$$2c_2^{\rm m} = k^2 c_0^{\rm H} - \frac{3}{2}\left(c_2^{\rm r} - \frac{3\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}c_1^{\rm r}\right) - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\left(c_1^{\rm m} - \frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}c_0^{\rm m}\right) \tag{11}$$

Since $c_0^{\rm r} = c_0^\theta = 0$ and the equations are linear, there are two varieties of intial conditions (i) $c_0^{\rm H} = 1$ and $c_0^{\rm m} = 0$ (ii) $c_0^{\rm H} = 0$ and $c_0^{\rm m} = 1$.

Case (i)

$$c_1^{\rm m} = -\frac{3}{2}c_1^{\rm r} \qquad c_1^{\rm r} = \frac{4}{3}c_1^{\rm m} \quad \to \quad c_1^{\rm m} = c_1^{\rm r} = 0 \quad \to \quad c_2^\theta = c_1^{\rm H} = 0\,. \tag{12}$$

$$2c_2^{\rm m} = k^2 - \frac{3}{2}c_2^{\rm r} \qquad c_2^{\rm r} = \frac{4}{3}c_2^{\rm m} \qquad c_3^\theta = \frac{1}{12}k^2 c_2^{\rm r}\,, \tag{13}$$

from which one can deduce that

$$c_2^{\rm m} = \frac{1}{4}k^2 \qquad c_2^{\rm r} = \frac{1}{3}k^2 \qquad c_3^\theta = \frac{1}{36}k^4\,. \tag{14}$$

and hence

$$\begin{aligned} \delta_{\rm m} &= \frac{1}{4}k^2\eta^2 + \mathcal{O}(\eta^3)\,, \\ \delta_{\rm r} &= \frac{1}{3}k^2\eta^2 + \mathcal{O}(\eta^3)\,, \end{aligned}$$

$$\begin{aligned}
\theta_{\rm r} &= \frac{1}{36}k^4\eta^3 + \mathcal{O}(\eta^4)\,, \\
H &= 1 - \frac{1}{36}k^2\eta^2 + \mathcal{O}(\eta^3)\,.
\end{aligned} \tag{15}$$

Case (ii)

$$c_1^{\rm m} = -\frac{3}{2}c_1^{\rm r} - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}} \qquad c_1^{\rm r} = \frac{4}{3}c_1^{\rm m} \quad \to \quad c_1^{\rm m} = -\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}} \qquad c_1^{\rm r} = -\frac{2\omega_{\rm m}}{3\sqrt{\omega_{\rm r}}}\,, \tag{16}$$

which implies that

$$c_2^{\theta} = -\frac{k^2\omega_{\rm m}}{12\sqrt{\omega_{\rm r}}} \qquad c_1^{\rm H} = -\frac{\omega_{\rm m}}{6\sqrt{\omega_{\rm r}}}\,. \tag{17}$$

and hence

$$\begin{aligned}
\delta_{\rm m} &= 1 - \frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta + \mathcal{O}(\eta^2)\,, \\
\delta_{\rm r} &= -\frac{2\omega_{\rm m}}{3\sqrt{\omega_{\rm r}}}\eta + \mathcal{O}(\eta^2)\,, \\
\theta_{\rm r} &= -\frac{\omega_{\rm m}}{12\sqrt{\omega_{\rm r}}}k^2\eta + \mathcal{O}(\eta^3)\,, \\
H &= \frac{\omega_{\rm m}}{6\sqrt{\omega_{\rm r}}}\eta + \mathcal{O}(\eta^2)\,.
\end{aligned} \tag{18}$$