EARLY UNIVERSE PHYS40772 2009 : Dr R.A. Battye

EXAMPLES SHEET 2 : STRUCTURE FORMATION

1. Correlation function

For the purposes of this question use Δ to denote the density contrast $\delta\rho/\rho$. If

$$\Delta(\mathbf{x}) = \sum_i \delta\left(\mathbf{x} - \mathbf{x}_i\right) , \tag{1}$$

show that the correlation function is given by

$$\xi(\mathbf{r}) = \frac{1}{V}\sum_{i,j} \delta\left(\mathbf{r} - \mathbf{x}_i + \mathbf{x}_j\right) . \tag{2}$$

Hence, deduce an expression for $|\Delta_k|^2$.

2. Decaying modes in the Mezaros equation

If $y = a/a_{\rm eq}$ then the matter density contrast $\delta_{\rm m}$ satisfies

$$\frac{d^2\delta_{\rm m}}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_{\rm m}}{dy} - \frac{3}{2y(1+y)}\delta_{\rm m} = 0\,, \tag{3}$$

in a universe containing matter and radiation. One solution is $\delta_{\rm m} = 2/3 + y$. By setting $\delta_{\rm m} = (2/3 + y)v(y)$ or otherwise derive another solution to the differential equation and show that it has the correct asymptotic behaviour for $y << 1$ and $y >> 1$.

3. Scale invariance

The variance on scales size R defined by window function $W(kR)$ is given by

$$\sigma_R^2 = \frac{V}{(2\pi)^3}\int d^3k P(k,\eta)|W(kR)|^2\,. \tag{4}$$

If the initial power spectrum is $P_{\rm i} \propto k^n$, compute $\sigma_{\rm R}$ at horizon crossing and show that it is independent of time if $n = 1$.

4. Power spectrum normalization

σ_8, the variance in the density fluctuation inside spheres of size $8h^{-1}$Mpc, is observed to be 0.6. Use the CDM power spectrum $P(k) = Ak$ for $k << k_{\rm eq}$ and $P(k) = Ak^{-3}$ for $k >> k_{\rm eq}$ to show that σ_R is constant for $R << R_{\rm eq}$ and hence compute A. What does this mean for $\sigma_{\rm R}$ when $R >> R_{\rm eq}$. What happens if the dark matter is hot?

5. Sound speed in matter and radiation universe

Consider a matter and radiation dominated universe. Derive an expression for the sound speed

$$c_{\rm s}^2 = \frac{dP}{d\rho}\,, \tag{5}$$

in terms of the ratio $\rho_{\rm m}/\rho_{\rm r}$. What is the value of the sound speed at $t_{\rm eq}$?

6. CDM transfer function

The growing solution of the Mezaros equation is $\delta_{\rm m} \propto 2/3 + y$ where $y = a/a_{\rm eq}$. Use this to show that the tranfer function can be approximated by

$$T(k) = C\left[1 + A\left(\frac{k}{k_{\rm eq}}\right) + B\left(\frac{k}{k_{\rm eq}}\right)^2\right]^{-1}, \tag{6}$$

where A, B and C are constants to be determined.

7. Manipulations of synchronous gauge EOM

Two of the synchronous gauge equations of motion are

$$k^2 H' = -\frac{2\omega_{\rm r}\theta_{\rm r}}{a^2}, \qquad \theta_{\rm r}' = \frac{1}{4}k^2\delta_{\rm r}\,. \tag{7}$$

Use these equations to show that

$$H'' + 2\frac{a'}{a}H' = -\frac{\omega_{\rm r}\delta_{\rm r}}{2a^2}\,. \tag{8}$$

8. Hot dark matter

The three neutrinos have standard hierachy where $m_{\nu_{\rm e}} >> m_{\nu_\mu} >> m_{\nu_\tau}$, and the difference between the electron and muon neutrinos is shown to be $(\Delta m)^2 = 2.5 \times 10^{-3}{\rm eV}^2$. Estimate the contribution of the massive neutrinos to the density of the universe? At what temperature will the electron neutrino become non-relativistic and what is the size of the horizon at this epoch?

9. Conformal Newtonian Gauge

Convert the synchronous gauge equation

$$k^2 H = \frac{a'}{2a}h' + \frac{3}{2}\left(\frac{\omega_{\rm r}\delta_{\rm r}}{a^2} + \frac{\omega_{\rm m}\delta_{\rm m}}{a}\right), \tag{9}$$

to the conformal Newtonian gauge. Which well known equation is does this resemble?