EARLY UNIVERSE PHYS40772 2009 : Dr R.A. Battye

EXAMPLES SHEET 1 : FRW UNIVERSE

1. Dynamical equations

Show that the one can derive the Raychauduri equation from

$$\dot{\rho} + 3H(\rho + P) = 0 \qquad \text{and} \qquad H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,.$$

2. Scale factor during radiation and matter domination

Compute an expression for the scale factor in terms of conformal time in universe containing only matter and radiation with densities relative to critical give by $\Omega_{\rm m}$ and $\Omega_{\rm r}$ respectively.

3. Natural units

Convert the following to natural units :

$$\text{(a) } 10^{15}M_{\odot} \quad \text{(b) } 500\mu\text{Jy} \quad \text{(c) } 1000\text{km}\,\text{sec}^{-1}\,.$$

Convert the Planck density $\approx 1.6\times 10^{57}\text{GeV}^4$ into CGS units.

If an object has a mass per unit length of $(10^{16}\text{GeV})^2$, what is this in SI units?

(NB $1\text{Jy} = 10^{-26}\text{W}\,\text{m}^{-2}\text{Hz}^{-1}$)

4. FRW timescale

Write down the Friedmann equation for a flat, homogeneous and isotropic universe containing a component with energy density ρ. What conservation equation does ρ satisfy? If the energy component is pressureless matter, how does ρ depend on the scale factor $a(t)$? If the scale factor is unity at the present day, show that

$$H_0 t_0 = \frac{2}{3}$$

where H_0 is the Hubble constant and t_0 is the age of the universe. Note that the critical density at the present day is given by $\rho_{\rm crit} = 3H_0^2/8\pi G$ where G is the gravitational constant.

5. Luminosity distance

Consider a flat universe which comprizes matter and a cosmological constant with densities relative to the critical density at the present day $\Omega_{\rm m}$ and Ω_Λ respectively. The luminosity distance of a source of radiation at redshift $1+z = 1/a$ is given by

$$d_{\rm L} = (1+z)\int_0^z \frac{dz'}{H(z')}\,,$$

where $H(z)$ is the Hubble parameter at redshift z. Show that

$$H(z) = H_0 \left[1 + \frac{3}{2}\Omega_{\rm m} z + \mathcal{O}(z^2)\right] ,$$

and hence deduce the coefficient of z^2 in a power law expansion of $d_{\rm L}$. Hence, deduce the value of q_0.

6. Horizon scale at Matter-Radiation Equality

Using the solutions for the scale factor of a flat universe during the matter-radiation transition, show that the conformal time at matter-radiation equality is given by $\eta_{\rm eq} = 2(\sqrt{2}-1)H_0^{-1}\Omega_{\rm m}^{-\frac{1}{2}}a_{\rm eq}^{\frac{1}{2}}$. Hence, show that the horizon at matter-radiation equality, when evolved as a comoving scale is today, is equal to $16\Omega_{\rm m}^{-1}h^{-2}$Mpc.

7. Multipole coefficients

The observed temperature anisotropies are decomposed in terms of spherical harmonics

$$\frac{\Delta T}{T} = \sum a_{lm} Y_{lm}(\theta,\phi) .$$

Use the orthogonality of spherical harmonics to show that

$$a_{lm} = \int d\Omega Y_{lm}^*(\theta,\phi)\frac{\Delta T}{T}(\theta,\phi) .$$

If the anisotropies are azimuthally symmetric

$$a_{l0} = \sqrt{(2l+1)\pi}\int_1^{-1} dx\, P_l(x)\frac{\Delta T}{T}(\cos^{-1}x)$$

and all the other multipole coefficients are zero.

8. Correlation function

For the purposes of this question use Δ to denote the density contrast $\delta\rho/\rho$. If

$$\Delta(\mathbf{x}) = \sum_i \delta\left(\mathbf{x}-\mathbf{x}_i\right) , \tag{1}$$

show that the correlation function is given by

$$\xi(\mathbf{r}) = \frac{1}{V}\sum_{i,j}\delta\left(\mathbf{r}-\mathbf{x}_i+\mathbf{x}_j\right) . \tag{2}$$

Hence, deduce an expression for $|\Delta_k|^2$.

9. Critical density in radiation and matter eras

Compute the Hubble parameter as a function of time in the radiation and matter dominated eras. Use this to deduce that the critical density is given by

$$\rho_{\rm crit} \approx \frac{3}{32\pi G t^2} ,$$

in the radiation era and

$$\rho_{\text{crit}} \approx \frac{1}{6\pi G t^2} \, .$$

in the matter era.

10. Time of EWPT

Show that the time, t, and the temperature, T,in the radiation era are related by

$$t \approx 0.3 g^{-1/2} \frac{m_{\text{pl}}}{T^2} \, ,$$

and use this to deduce the cosmic time at which the Electroweak phase transition takes place.