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Golden Equations for Lectures 5 to 8

$$
\begin{aligned}
\hat{A}|v\rangle = |Av\rangle \quad &\Rightarrow \quad \langle Av| = \langle v|\hat{A}^\dagger \qquad \text{for all } |v\rangle \\
\text{Hermitian:} \quad & \quad \hat{A} = \hat{A}^\dagger \\
\text{Unitary:} \quad & \quad \hat{U}^\dagger = \hat{U}^{-1} \\
\hat{A}|a_i\rangle \quad &= \quad a_i|a_i\rangle
\end{aligned}
$$

The Stern-Gerlach Experiment

The experiment that revealed the quantisation of spin was done by Otto Stern and Walther Gerlach in Frankfurt in 1921. See handout IQM9 from last term's PHYS 20101 for a basic description, or the first part of Chapter 1 of Townsend's book for more details. Richard Feynman invented a classic series of thought-experiments that use variants on the Stern-Gerlach experiment (SGE) to reveal the weirdness of quantum mechanics while keeping the maths extremely simple: this is the model for the rest of Townsend's first chapter, and some of the material at the start of Section 2 of this course comes from there.

The SGE changes the direction of particles according to the component of the magnetic moment, $\boldsymbol{\mu}$, that is aligned with its magnetic field; in simple cases (e.g. silver atoms) $\boldsymbol{\mu}$ is aligned with spin. Feynman's idea was to pass a beam of atoms through several SGE one after another, and look at the effect of changing the relative directions of their magnets. You can try this for yourself by playing with one or both of the **on-line SG simulators** linked from the course web site...strongly recommended!

Examples

Lecture 9 Wave mechanics vs. matrix mechanics?

1. Revision of key concepts from PHYS2010:
   Let operator $\hat{H}$ describe the energy of a one-particle system, which has eigenvalues $E_1, E_2, \ldots$ (all different), corresponding to eigenfunctions $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots$.
   At some instant the wavefunction of the system is a superposition:

$$\psi(\mathbf{x}) = c_1\,\phi_1(\mathbf{x}) + c_2\,\phi_2(\mathbf{x}).$$

   (a) Evaluate $\hat{H}\psi(\mathbf{x})$.

   (b) Write down the physical dimensions of each of $\hat{H}, E_1, c_1, \phi_1(\mathbf{x}), \psi(\mathbf{x}), \mathbf{x}$. (Warning: think hard about the wave functions!).

   (c) If a measurement of the energy is made, what are the possible values that might be found, and what are the probabilities of each result?

   (d) If a measurement of position is made, write down the formula for the probability that the particle is within 1 nm of the centre of the coordinate system.

2. Two observables for a particle with state space $V^3(\mathbb{C})$ are represented in a certain basis by the operators:

$$L_x \longrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad L_z \longrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   (a) What are the possible values one can obtain if L_z is measured?

   (b) Take the state in which $L_z = \hbar$. In this state, what are $\langle L_x\rangle$, $\langle L_x^2\rangle$, and ΔL_x?

   (c) Find the normalised eigenstates and the eigenvalues of L_x in the L_z basis.

   (d) If the particle is in the state with $L_z = -\hbar$, and L_x is measured, what are the possible outcomes and their probabilities?

   (e) Consider the state

$$|\psi\rangle \underset{L_z}{\longrightarrow} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix},$$

   i.e. this is the representation in the L_z basis. If L_x^2 is measured in this state and a result $\hbar^2$ is obtained, what is the state after the measurement? How probable was this result? If L_z is then measured, what are the outcomes and respective probabilities?

(f) A particle is in a state for which the probabilities are $\text{Prob}(L_z = \hbar) = 1/4$, $\text{Prob}(L_z = 0) = 1/2$, $\text{Prob}(L_z = -\hbar) = 1/4$. Convince yourself that the most general normalised state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = \hbar\rangle + \frac{e^{i\delta_2}}{\sqrt{2}}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -\hbar\rangle.$$

In the lectures I said that if $|\psi\rangle$ is a normalised state then $e^{i\theta}|\psi\rangle$ represents the same physical state. Does this mean that the factors $e^{i\delta_i}$ multiplying the L_z eigenstates are irrelevant? Check by calculating $\text{Prob}(L_x = 0)$.

Lecture 10

1. Show that

$$e^x = \lim_{N\to\infty}\left[1 + \frac{x}{N}\right]^N,$$

by comparing the Maclaurin series expansions for the two functions.

2. From the abstract Schrödinger equation and the definition of the time evolution operator $\hat{U}(t_0, t)$, show that

(a)

$$i\hbar\frac{d\hat{U}(t_0,t)}{dt} = \hat{H}(t)\,\hat{U}(t_0,t)$$

(b)

$$i\hbar\frac{d\hat{U}(t_0,t)}{dt_0} = -\hat{H}(t_0)\,\hat{U}(t_0,t)$$

(c) Hence, if $\hat{U}(t_0,t) = \hat{U}(t - t_0)$ (e.g. for a closed system) then show that $\hat{H}(t) = \hat{H}(t_0)$, i.e. $\hat{H}$ is independent of time.

3. Let

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

with $E_1 \neq E_0$, represent the Hamiltonian for a three-state system in the "crispies" representation, where the basis states are:

$$|\text{snap}\rangle \xrightarrow[\text{crispies}]{} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |\text{soggy}\rangle \xrightarrow[\text{crispies}]{} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad |\text{pop}\rangle \xrightarrow[\text{crispies}]{} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

.

(a) If the state of the system at time $t = 0$ is $|\text{soggy}\rangle$, what is the state at time t?

(b) If the starting state is $|\text{snap}\rangle$, what happens?

Lecture 11

1. To show that probability amplitudes must be complex numbers in general:

   (a) By considering Stern-Gerlach experiments as discussed in the lectures, but with axes oriented along (z and y) or (x and y) instead of (z and x), convince yourself that a ket which is "spin-up" in the y direction must be of the form

   $$|+y\rangle = \frac{e^{i\gamma_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\gamma_-}}{\sqrt{2}}|-z\rangle$$

   where γ_+ and γ_- are yet-to-be determined phases; also that

   $$|\langle +y|+x\rangle|^2 = \frac{1}{2}.$$

   Here $|+z\rangle$ is spin-up along the z axis etc, i.e. what we usually call $|\uparrow\rangle$, as we have run out of arrow symbols now we have to include y as well!

   (b) By combining $\langle +y|$ with $|+x\rangle = (|+z\rangle + |-z\rangle)/\sqrt{2}$, show that

   $$|\langle +y|+x\rangle|^2 = \frac{1}{2}(1 + \cos(\gamma_+ - \gamma_-)),$$

   Hence show that $\langle +z|+y\rangle$ and $\langle -z|+y\rangle$ cannot both be real numbers.

2. From their effects on the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, show that for a system with angular momentum $\hbar/2$:

   (a) the z-spin operator can be written:

   $$\hat{S}_z = (\hbar/2)(\hat{P}_\uparrow - \hat{P}_\downarrow),$$

   where $\hat{P}_\uparrow$ and $\hat{P}_\downarrow$ are the projectors onto the spin-up and spin-down states respectively.

   (b) the spin ladder operators can be written:

   $$\hat{S}_+ = \hbar|\uparrow\rangle\langle\downarrow| \quad \text{and} \quad \hat{S}_- = \hbar|\downarrow\rangle\langle\uparrow|$$

3. Given the formula for ladder operators:

   $$\hat{J}_\pm|j,m\rangle = c_\pm(j,m)|j,m\pm 1\rangle,$$

   with

   $$c_\pm(j,m) = \sqrt{j(j+1) - m(m\pm 1)}\,\hbar,$$

   find the 5×5 matrices for J_+ and J_- for the case $j = 2$; hence construct J_x and J_y (this is easier than it sounds as only one diagonal is non zero in each of $J_\pm$).

Lecture 12

1. Consider a direct product space $V_{1\otimes 2} = V_1 \otimes V_2$. Let the product-space operator equivalent to Ω_1 (on V_1, as indicated by subscript) be $\Omega_1^{1\otimes 2} = \Omega_1 \otimes I$, and so on. Show that
   (a) $(\Omega \otimes \Gamma)(\Theta \otimes \Lambda) = (\Omega\Theta) \otimes (\Gamma\Lambda)$;
   (b) $A \otimes I + B \otimes I = (A + B) \otimes I$;
   (c) Operators on the factor spaces commute, i.e. $[\Omega_1^{1\otimes 2}, \Lambda_2^{1\otimes 2}] = 0$;
   (d) If $[\Omega_1, \Lambda_1] = \Gamma_1$ on V_1, then $[\Omega_1^{1\otimes 2}, \Lambda_1^{1\otimes 2}] = \Gamma_1^{1\otimes 2}$;
   (e) $\left(\Omega_1^{1\otimes 2} + \Omega_2^{1\otimes 2}\right)^2 = (\Omega_1^2) \otimes I + I \otimes (\Omega_2^2) + 2\Omega_1 \otimes \Omega_2$.

HINTS:

Lecture 9: 2a You don't need to solve any equations to answer this! Lecture 10: 3. You have to find the eigenvalues and eigenvectors of the Hamiltonian matrix. Then express the original basis vectors as superpositions of energy eigenvectors. Then you can use the simple rule for how energy eigenstates change with time to work out how the states change. Lecture 12: 1(a) and (b): Consider the action of the operator on a simple direct product ket $|v\rangle \otimes |w\rangle$. Argue using linearity that if two operators have the same effect on all simple product kets, they will have the same effect on any arbitrary ket in $V_{1\otimes 2}$ (note that most such kets are not simple direct products). If two operators have the same effect on all kets, they are equal.

Answers to Handout 2

Lecture 5

1.

$$|a\rangle = \sum_i |i\rangle\langle i|a\rangle = \hat{I}|a\rangle$$

$|a\rangle$ is an arbitrary ket. Integer i indexes the dimensions of the vector space. The set $\{|i\rangle\}_{i=1}^{N}$ is an orthonormal basis. $\hat{I}$ is the identity operator. This equation says that (i) an arbitrary ket can be expanded in terms of an orthonormal basis, with coefficient (or coordinate) of the ith basis vector being $\langle i|a\rangle$; (ii) It identifies the identity operator with the sum over the projectors onto all the basis vectors, i.e. $\sum |i\rangle\langle i| = \hat{I}$.

$$\left[\langle b|\hat{A}\right](|a\rangle) = \langle b|\left(\hat{A}|a\rangle\right) \equiv \langle b|\hat{A}|a\rangle$$

$|a\rangle$, $\langle b|$ and $\hat{A}$ are an arbitrary ket, bra and linear operator, respectively. This says that the matrix element $\langle b|\hat{A}|a\rangle$ can be read either as the action of bra $\langle b|$ on the ket resulting from working $\hat{A}$ to the right on ket $|a\rangle$, or as the action on ket $|a\rangle$ of the bra produced by working $\hat{A}$ to the left on bra $\langle b|$—either approach is guaranteed to give the same numerical answer.

$$\langle a|\hat{B}|b\rangle = (\langle a|1\rangle,\ \langle a|2\rangle) \begin{pmatrix} \langle 1|\hat{B}|1\rangle & \langle 1|\hat{B}|2\rangle \\ \langle 2|\hat{B}|1\rangle & \langle 2|\hat{B}|2\rangle \end{pmatrix} \begin{pmatrix} \langle 1|b\rangle \\ \langle 2|b\rangle \end{pmatrix} \equiv [a]^{T*}[B][b]$$

$\langle a|$ and $|b\rangle$ are an arbitrary bra and ket respectively in a 2-dimensional vector space, and $\hat{B}$ is an operator in that space. $\{|1\rangle, |2\rangle\}$ is an orthonormal basis in the space and $\langle 1|, \langle 2|$ are the bras corresponding to the basis kets. This equation shows how to evaluate an abstract "matrix element" such as $\langle a|\hat{B}|b\rangle$ by evaluating the matrix representations of bras, operators and kets in a particular basis, so the final number can be found by matrix multiplication. It also reminds you of our square-bracket notation to show when we are talking about 'concrete' matrices (dependent on a particular basis) versus the basis-independent abstract Dirac notation.

2. Let $\hat{U}_1$, $\hat{U}_2$ be unitary operators, and $\hat{V} = \hat{U}_1\hat{U}_2$. Then $\hat{V}^\dagger\hat{V} = (\hat{U}_1\hat{U}_2)^\dagger(\hat{U}_1\hat{U}_2) = \hat{U}_2^\dagger\hat{U}_1^\dagger\hat{U}_1\hat{U}_2 = \hat{U}_2^\dagger\hat{I}\hat{U}_2 = \hat{U}_2^\dagger\hat{U}_2 = \hat{I}$. Therefore $\hat{V}$ is unitary.

3. (i) According to the question we start with an $N \times N$ matrix (say $[U]$) whose columns are orthonormal vectors. Note that this implies we are dealing with an N-dimensional vector space. Since there are N columns, there are N such vectors, and any N orthonormal vectors in an N-D space form a basis; let's call

this one $\{|y_i\rangle\}$; call the original basis $\{|x_i\rangle\}$. Our matrix components, and hence the components of $|y_i\rangle$, are

$$U_{ji} \equiv \langle x_j|\hat{U}|x_i\rangle = [y_i]_j = \langle x_j|y_i\rangle.$$

(notice that, as always, the column index (here i) comes second). The rows are $\langle x_j|y_i\rangle$ for fixed j, i.e. the complex conjugate of the components of $|x_j\rangle$ in the $|y_i\rangle$ basis. These are indeed orthonormal:

$$\sum_i \langle x_j|y_i\rangle^* \langle x_k|y_i\rangle = \sum_i \langle x_k|y_i\rangle\langle y_i|x_j\rangle = \langle x_k|\hat{I}|x_j\rangle = \langle x_k|x_j\rangle = \delta_{kj}.$$

An identical argument applies if we start with orthonormal rows and wish to prove that the columns are orthonormal.

(ii) To prove $[U]$ is unitary:

$$[U^\dagger U]_{ik} = \sum_j (U_{ij})^\dagger U_{jk} = \sum_j U^*_{ji} U_{jk} = \sum_j \langle x_j|y_i\rangle^* \langle x_j|y_k\rangle = \sum_j \langle y_i|x_j\rangle\langle x_j|y_k\rangle = \delta_{ik}.$$

Lecture 6

1. (a) $(\hat{U}\hat{A}\hat{U}^\dagger)^\dagger = (\hat{U}^\dagger)^\dagger \hat{A}^\dagger \hat{U}^\dagger = \hat{U}\hat{A}\hat{U}^\dagger$ therefore $\hat{U}\hat{A}\hat{U}^\dagger$ is Hermitian. Similarly $(\hat{U}^\dagger\hat{A}\hat{U})^\dagger = \hat{U}^\dagger\hat{A}^\dagger(\hat{U}^\dagger)^\dagger = \hat{U}^\dagger\hat{A}\hat{U}$.

   (b) To show $\{|b_i\rangle\}$ are an orthonormal basis in V^N, we require that (i) they are orthonormal:

   $$\langle b_i|b_j\rangle = \langle a_i|\hat{U}\hat{U}^\dagger|a_j\rangle = \langle a_i|\hat{I}|a_j\rangle = \langle a_i|a_j\rangle = \delta_{ij} \quad \text{QED}$$

   (ii) there are N members: True, since from (i) there is one $|b_i\rangle$ for each $|a_i\rangle$, and $\{|a_i\rangle\}$ is a basis). QED.

   (c) $\det([U][A][U]^\dagger) = \det([U])\det([A])\det([U]^\dagger)$, since the determinant of a product is the product of determinants. But from Theorem 1.9 we know that $\det([U])\det([U]^\dagger) = 1$, so $\det([U][A][U]^\dagger) = \det([A])$ QED.
   NB considered as a passive transform, unitary transforms like $\hat{U}\hat{A}\hat{U}^\dagger$ change the ortho-normal basis in which the operator $\hat{A}$ is represented; in this case from $\{|a_i\rangle\}$ to $\{|b_i\rangle\}$, since

   $$\langle a_i|\hat{U}\hat{A}\hat{U}^\dagger|a_j\rangle = \langle b_i|\hat{A}|b_j\rangle.$$

   Hence this result shows that the determinant is a property of the abstract operator and not just of a particular basis-dependent matrix representation.

2. (a) The positive x axis is not a subspace because if you multiply a vector on it by a negative scalar, e.g. $-1\mathbf{i}$, you get a vector which is not on the positive axis. Therefore this set is not closed under scalar multiplication, therefore not a vector space, therefore not a subspace.

(b) The plane $z = 1$ in $V^3(\mathbb{R})$ is not a subspace because it is not closed: if we add two vectors on this plane we get a vector on the plane $z = 2$, e.g.:

$$(1, 2, 1) + (3, 0, 1) = (4, 2, 2).$$

NB: in a vector space, all vectors start from the origin! (This is another way of saying that the zero vector is unique; therefore all subspaces must include the zero vector.)

3. We can always create a basis on V^N which includes any given vector (except 0): if the vector is not in a basis already, expand it:

$$|v\rangle = \sum_i v_i |x_i\rangle$$

then remove any one of the $|x_i\rangle$ with non-zero coefficients and replace with $|v\rangle$ — we still have a basis (N linearly-independent vectors), i.e. $\{|v\rangle, |x_1\rangle, |x_2\rangle \ldots |x_{N-1}\rangle\}$. If this is not already orthonormal we can use Gram-Schmidt, starting with $|v\rangle$. It follows that $\{|x_i\rangle\}_{i=1}^{N-1}$ are an orthonormal set, and all orthogonal to $|v\rangle$. Therefore any vector in the subspace spanned by the $\{|x_i\rangle\}$ (call it $V_\perp^{N-1}$) is orthogonal to $|v\rangle$. Furthermore, any vector $|w\rangle$ orthogonal to $|v\rangle$ is in $V_\perp^{N-1}$, because any vector can be written

$$|w\rangle = a_0|v\rangle + \sum_{i=1}^{N-1} a_i |x_i\rangle$$

using our basis on V^N. But if $\langle v|w\rangle = a_0 = 0$, we have

$$|w\rangle = \sum_{i=1}^{N-1} a_i |x_i\rangle$$

and therefore in $V_\perp^{N-1}$.

4. (a) $a_n^* \langle a_n| = \langle a_n|\hat{A}^\dagger$

(b) If $\hat{A}$ is Hermitian, $\hat{A}^\dagger = \hat{A}$, so:

$$\begin{aligned}
\langle a_j|\hat{A}|a_k\rangle &= \langle a_j| \left(\hat{A}|a_k\rangle\right) = \langle a_j| \left(a_k|a_k\rangle\right) = a_k\langle a_j|a_k\rangle \\
\langle a_j|\hat{A}|a_k\rangle &= \left(\langle a_j|\hat{A}\right)|a_k\rangle = \left(\langle a_j|\hat{A}^\dagger\right)|a_k\rangle = a_j^*\langle a_j|a_k\rangle \\
a_k\langle a_j|a_k\rangle &= a_j^*\langle a_j|a_k\rangle \Rightarrow (a_k - a_j^*)\langle a_j|a_k\rangle = 0
\end{aligned}$$

First consider the case when $k = j$. As noted in the lectures the zero vector does not count as an eigenket, so $\langle a_j|a_j\rangle \neq 0$. Therefore $a_j = a_j^*$, i.e. $a_j \in \mathbb{R}$—eigenvalues are real. Then, for $k \neq j$, if $a_j \neq a_k$ then we must have $\langle a_j|a_k\rangle = 0$ i.e. eigenkets with different eigenvalues are orthogonal.

(c) If $\hat{A}$ is unitary, $\hat{A}^\dagger = \hat{A}^{-1}$.

$$\begin{aligned}\langle a_j|\hat{A}^\dagger\hat{A}|a_k\rangle &= \left(\langle a_j|\hat{A}^\dagger\right)\left(\hat{A}|a_k\rangle\right) = a_j^* a_k\langle a_j|a_k\rangle \\ \langle a_j|\hat{A}^\dagger\hat{A}|a_k\rangle &= \langle a_j|\hat{I}|a_k\rangle = \langle a_j|a_k\rangle\end{aligned}$$

Taking $k = j$, since $\langle a_j|a_j\rangle \neq 0$, we must have $a_j^* a_j = 1$, i.e. a_j is a complex number of unit modulus and so can be written as $e^{i\theta_j}$, with $\theta_j \in \mathbb{R}$. If $k \neq j$ and $a_k \neq a_j$ then the kets are orthogonal as in the previous case.

Lecture 7

1. (a) Since both matrices are real, they will be symmetric if they are Hermitian. By inspection, $[A]$ is not and $[B]$ is.

   (b) Eigenvalues of $[A]$:

$$\det(A - aI) = \begin{vmatrix} 1-a & 3 & 1 \\ 0 & 2-a & 0 \\ 0 & 1 & 4-a \end{vmatrix} = (1-a)(2-a)(4-a) - 3\times 0 + 1\times 0.$$

Hence the eigenvalues are 1, 2, 4. Eigenvectors:

$$\begin{pmatrix} 1-a_n & 3 & 1 \\ 0 & 2-a_n & 0 \\ 0 & 1 & 4-a_n \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

For $a_1 = 1$ this gives $3y + z = 0$, $y = 0$, $y + 3z = 0$, hence, $y = z = 0$ and $x = 1$ by normalisation. For $a_2 = 2$ we get $-x + 3y + z = 0$, $0 = 0$, $y + 2z = 0$; so $y = -2z$ and $x = -5z$. For $a_3 = 4$ we get $-3x + 3y + z = 0$, $-2y = 0$, $y = 0$, so $z = 3x$. The normalised eigenvectors are

$$[a_1] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad [a_2] = \frac{1}{\sqrt{30}}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}, \quad [a_3] = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

If this is defined on a complex vector space all eigenvectors can be multiplied by an arbitrary phasor $e^{i\phi_n}$ without changing the normalisation. For a real vector space this degenerates to the option of multiplying by -1.

Eigenvalues of [B]:

$$\begin{aligned}\det(B - bI) &= \begin{vmatrix} 2-b & 1 & 1 \\ 1 & -b & -1 \\ 1 & -1 & 2-b \end{vmatrix} \\ &= (2-b)[(-b)(2-b) - 1] - [(2-b) + 1] + [-1 - (-b)] \\ &= (2-b)(b^2 - 2b - 1) - (3-b) - (1-b) \\ &= (2-b)(b^2 - 2b - 1) - 2(2-b) \\ &= (2-b)(b^2 - 2b - 3) = (2-b)(b+1)(b-3)\end{aligned}$$

so eigenvalues are $-1, 2, 3$. Solving for the eigenvectors:

$$(2-b)x + y + z = 0$$

$$x - by - z = 0$$

$$x - y + (2-b)z = 0$$

Adding the first and last line we get $(3-b)(x+z) = 0$, so $x = -z$ unless $b = 3$. For $b_1 = -1$ we have $y = z - x = 2z$. For $b_2 = 2$ we have $x = y = -z$. For $b_3 = 3$ we have from adding the first and second lines $-2y = 0$, so $x - z = 0$. Hence the normalized eigenvectors (up to a phase factor) are

$$[b_1] = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad [b_2] = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad [b_3] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) By inspection, the eigenvectors of $[A]$ are not orthogonal (in fact no pair of them are orthogonal), as expected for a non-Hermitian matrix. Actually $N \times N$ matrices which are neither Hermitian nor unitary may not even have N different eigenvectors. As expected for a Hermitian matrix, the eigenvectors of $[B]$ are orthogonal, for instance

$$\langle b_1 | b_2 \rangle = (-1 \times 1 + 2 \times 1 + 1 \times (-1))/\sqrt{18} = 0.$$

(d) This part only applies to $[B]$ as $[A]$ is not Hermitian. The matrix of eigenvectors is

$$[U] = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{3} \\ 2 & \sqrt{2} & 0 \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix}.$$

$$\begin{aligned}
[U]^\dagger[B][U] &= \frac{1}{6}\begin{pmatrix} -1 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{3} \\ 2 & \sqrt{2} & 0 \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} -1 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 2\sqrt{2} & 3\sqrt{3} \\ -2 & 2\sqrt{2} & 0 \\ -1 & -2\sqrt{2} & 3\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.
\end{aligned}$$

2. You should recognise $[R]$ as the standard 2D rotation matrix by an angle ϕ. So it must be unitary because its inverse is a rotation by $-\phi$ which gives the transpose of $[R]$, equivalent to the adjoint since all elements are real.

(i) To check unitarity explicitly:

$$[R]^\dagger[R] = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) Eigenvalues: characteristic equation is

$$\det(R - rI) = (\cos\phi - r)^2 + \sin^2\phi = 1 - 2r\cos\phi + r^2 = 0$$

Using the usual solution to quadratic equations:

$$r = \frac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2} = \cos\phi \pm \sqrt{-\sin^2\phi} = \cos\phi \pm i\sin\phi = e^{\pm i\phi}.$$

(iii) Eigenvectors: we have $x\cos\phi + y\sin\phi = (\cos\phi \pm i\sin\phi)x$, $y\cos\phi - x\sin\phi = (\cos\phi \pm i\sin\phi)y$. The first gives $y = \pm ix$, the second $-x = \pm iy$ which is the same thing. Hence normalized eigenvectors are

$$|{+\phi}\rangle \to \frac{e^{i\theta_+}}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |{-\phi}\rangle \to \frac{e^{i\theta_-}}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Orthogonality:

$$\langle +\phi | -\phi \rangle = e^{i(\theta_- - \theta_+)}(1^* \times 1 + i^* \times (-i)) = e^{i(\theta_- - \theta_+)}(1 + (-i)^2) = 0$$

(iv) Diagonalisation. Choosing the phases $\theta_+ = \theta_- = 0$,

$$[U] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$[U]^\dagger[R][U] = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\phi + i\sin\phi & \cos\phi - i\sin\phi \\ i\cos\phi - \sin\phi & -\sin\phi - i\cos\phi \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2(\cos\phi + i\sin\phi) & 0 \\ 0 & 2(\cos\phi - i\sin\phi) \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Lecture 8

1. (i) The characteristic equation is

$$0 = \det(C - cI) = \begin{vmatrix} 1-c & 0 & 1 \\ 0 & -c & 0 \\ 1 & 0 & 1-c \end{vmatrix}$$
$$= (1-c)(-c)(1-c) - 1 \times (-c) = c(1 - (1-c)^2)$$
$$= c(1 - 1 + 2c - c^2) = c^2(2 - c),$$

hence eigenvalues are $0, 0, 2$ as required. (ii) Solving for the eigenvector for $c_3 = 2$, we have $-x + z = 0$, $2y = 0$, $x - z = 0$, i.e. $y = 0$ and $x = z$ (twice). Hence

$$|c_3\rangle \to N \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

where the normalisation constant N is defined by $N^* N \times 2 = 1$, hence $N = e^{i\theta}/\sqrt{2}$ for any $\theta \in \mathbb{R}$. (iii) Solving for $c = 0$ we get $x + z = 0$, $0 = 0$, $x + z = 0$, so $x = -z$, and y is unconstrained. Hence vectors are of the form

$$N \begin{pmatrix} f \\ g \\ -f \end{pmatrix}$$

with $N^2 \sqrt{|g|^2 + 2|f|^2} = 1$, as required. (No phase factor, since it is absorbed into the unknowns f and g). Alternatively, requiring that a vector in the $c = 0$ subspace be orthogonal to $|c_3\rangle$ gives

$$\langle c_3 | v \rangle = (1 \times x + 0 \times y + 1 \times z)/\sqrt{2} = 0$$

which gives the same result.

2. Commutator:

$$\begin{aligned}
[C, B] &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} = 0
\end{aligned}$$

Diagonalising $[C]$ using the matrix of eigenvalues for $[B]$ derived in Q8.1 gives

$$\begin{aligned}
[U]^\dagger [C][U] &= \frac{1}{6} \begin{pmatrix} -1 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & \sqrt{2} & \sqrt{3} \\ 2 & \sqrt{2} & 0 \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} -1 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\end{aligned}$$

which has the eigenvalues $0, 0, 2$ on the diagonal as it should. Hence $|c_3\rangle = |b_3\rangle$ and the $c = 0$ subspace is spanned by $|b_1\rangle$ and $|b_2\rangle$.

3. (a) By definition,

$$e^{A+B} = \sum_{m=0}^{\infty} \frac{(A+B)^m}{m!}.$$

With $[A, B] = 0$, we can simplify operator product terms like $A^a B^b A^c B^d$ into $A^{a+c} B^{b+d}$ (and so on), and so we can use the binomial theorem to get

$$e^{A+B} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} \frac{A^{m-n} B^n}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{A^{m-n} B^n}{(m-n)! n!}$$

Meanwhile,

$$e^A e^B = \left(\sum_{m=0}^{\infty} \frac{A^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A^m B^n}{m! n!}$$

The double sums at the end of our two expressions are in fact equal, since each counts every pair of positive powers of A and B once and only once, and the denominators are the same in each case (the first expression effectively sums the diagonals of the matrix $M_{ij} = A^i B^j / i! j!$, then sums the results, while the second sums the columns, then sums the results).

(b) The adjoint of e^{iA} is

$$(e^{iA})^\dagger = \sum_{n=0}^{\infty} \frac{[(iA)^n]^\dagger}{n!} = \sum_{n=0}^{\infty} \frac{(-iA^\dagger)^n}{n!} = e^{-iA^\dagger} = e^{-iA}.$$

The second equality uses the rules that (i) complex numbers go to their conjugates on taking the adjoint (ii) $(A^{n-1}A)^\dagger = A^\dagger (A^{n-2}A)^\dagger$ etc to show that $(A^n)^\dagger = (A^\dagger)^n$. At the end we use $A = A^\dagger$, since A is Hermitian. But since A commutes with any scalar multiple of itself,

$$e^{-iA} e^{iA} = e^{-i(A-A)} = e^0 = I,$$

where we write I and not 1 since this is an operator equation. Similarly $e^{iA} e^{-iA} = e^{i(A-A)} = I$. Hence the adjoint of e^{iA} is its inverse, so it is unitary.