1. For *each* of the following expressions, say whether it is (or evaluates to) a ket, a bra, a complex number, or an operator:

   (i) $\langle\psi|\hat{L}_z$, (ii) $|1\rangle\langle 1|$, (iii) $|f\rangle \otimes |h\rangle$, (iv) $\langle\uparrow|\leftarrow\rangle$.

2. Given that
$$|\leftarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}},$$
   where $\{|\uparrow\rangle, |\downarrow\rangle\}$ are the eigenstates of $\hat{S}_z$, evaluate the 2×2 matrix representing the operator $|\leftarrow\rangle\langle\leftarrow|$ in the S_z basis. What kind of operator is this?

3. (i) The following equations are written rather 'loosely': re-write in a form that is fully consistent:
$$\hat{M} = \hat{S}_z^{\;2} + 2\hbar^2$$
$$\hat{J} = \hat{L} + \hat{S}$$
   (ii) In the equation
$$\hat{L}_z|m = 0\rangle = 0,$$
   what does the symbol '0' on the right-hand side stand for?

4. **Part of 2010 Exam Q2:** A certain ion in a crystal has a Hamiltonian $\hat{H} = A\hat{L}_x^2$, where $\hat{L}_x$ is the x-component of orbital angular momentum and A is a positive constant. The ion has angular momentum quantum number $l = 1$. In the L_z representation, this gives
$$\hat{H} \underset{L_z}{\longrightarrow} \frac{A\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$
   (a) Find all the eigenvalues of $\hat{H}$, and the normalised eigenvector of the ground state (only).

   (b) The ion's state at time $t = 0$ is $|m = -1\rangle$. Analyse this into a component in the ground state and a component orthogonal to the ground state, and hence find the state at a later time t.

5. **2009 Exam Q1(c)** A hydrogen atom is in the state
$$|\psi\rangle = N(|1,0,0\rangle + 3|2,1,-1\rangle + 2|3,2,2\rangle).$$
   where $|n,l,m\rangle$ is a hydrogenic state with the usual quantum number notation. If the z component of orbital angular momentum is measured, give the possible results with their probabilities of occurence. [6 marks]

1. (i) Bra (operator acts to left on a bra to make a new bra).

   (ii) Operator (outer products are operators; this is the projector to state $|1\rangle$).

   (iii) Ket (direct product of two kets is a ket in the direct product space).

   (iv) Complex numbers (inner products are complex numbers).

2. $|{\leftarrow}\rangle\langle{\leftarrow}|$ is the projection operator onto state $|{\leftarrow}\rangle$; call it P. First basis ket is $|{\uparrow}\rangle$, second is $|{\downarrow}\rangle$, so

$$\begin{aligned} P_{11} &= \langle\uparrow|P|\uparrow\rangle = \langle\uparrow|{\leftarrow}\rangle\langle{\leftarrow}|\uparrow\rangle = \frac{1}{2}\langle\uparrow|\left(|\uparrow\rangle + |\downarrow\rangle\right)\left(\langle\uparrow| + \langle\downarrow|\right)|\uparrow\rangle \\ &= \frac{1}{2}\left(\langle\uparrow|\uparrow\rangle + \langle\uparrow|\downarrow\rangle\right)\left(\langle\uparrow|\uparrow\rangle + \langle\downarrow|\uparrow\rangle\right) = \frac{1}{2}(1+0)(1+0) = \frac{1}{2} \end{aligned}$$

   using the orthonormality of $\{|\uparrow\rangle, |\downarrow\rangle\}$. Similarly:

$$\begin{aligned} P_{12} &= \langle\uparrow|P|\downarrow\rangle = \langle\uparrow|{\leftarrow}\rangle\langle{\leftarrow}|\downarrow\rangle = \frac{1}{2}(1+0)(0+1) = \frac{1}{2} \\ P_{21} &= \langle\downarrow|P|\uparrow\rangle = \langle\downarrow|{\leftarrow}\rangle\langle{\leftarrow}|\uparrow\rangle = \frac{1}{2}(0+1)(1+0) = \frac{1}{2} \\ P_{22} &= \langle\downarrow|P|\downarrow\rangle = \langle\downarrow|{\leftarrow}\rangle\langle{\leftarrow}|\downarrow\rangle = \frac{1}{2}(0+1)(0+1) = \frac{1}{2} \end{aligned}$$

   So the matrix

$$[P] = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

   This is the one we met in the last example class.

3. (i)

$$\hat{M} = \hat{S}_z^{\,2} + 2\hbar^2$$

   This is an operator equation so all terms should be operators. However, since multiplying by the identity has no effect we often don't bother writing $\hat{I}$; but to be accurate we should say

$$\hat{M} = \hat{S}_z^{\,2} + 2\hbar^2\hat{I}.$$

$$\hat{J} = \hat{L} + \hat{S}$$

   J is an operator that acts on the combined (direct product) space of orbital angular momentum and spin, but $\hat{L}$ and $\hat{S}$ act only on one (each) of these factor spaces. To be precise we should write

$$\hat{J} = \hat{L}\otimes\hat{I} + \hat{I}\otimes\hat{S}.$$

(ii) In the equation

$$\hat{L}_z|m=0\rangle = 0,$$

the '0' on the right-hand side is the zero vector (zero ket), since the LHS (operator acting on ket) evaluates to a ket, not to a scalar.

4. (a) Given that we know $H = AL_x^2$, which is a function of L_x, its eigenvalues will be the same functions of the L_x eigenvalues. L_x must have the same eigenvalues as L_z, since x and z are just conventional labels. For $l = 1$, the eigenvalues of L_z and L_x are $\hbar$, 0, and $-\hbar$. Hence we expect eigenvalues of H to be

$$A\hbar^2 \times (1,0,1)$$

Check by explicitly solving the matrix. For speed, solve the matrix without the pre-factor (call it H') and then multiply the eigenvalues by $A\hbar^2/2$:

$$\begin{aligned}
\det(H' - mI) &= \begin{vmatrix} 1-m & 0 & 1 \\ 0 & 2-m & 0 \\ 1 & 0 & 1-m \end{vmatrix} \\
&= (1-m)[(2-m)\times(1-m) - 0] - 0 + 1 \times [0 - (2-m)\times 1] \\
&= (2-m)[1 - 2m + m^2 - 1] = (2-m)m(m-2),
\end{aligned}$$

i.e. $m = (2,0,2)$ and hence energy eigenvalues are $E = A\hbar^2(1,0,1)$ as expected.

The ground state is the lowest energy value, in this case $E = 0$. Hence the eigenvector satisfies:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0,$$

i.e. $a + c = 0$ and $2b = 0$. Hence the normalised ground state ket is

$$|E=0\rangle \xrightarrow[L_z]{} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(b) The state with $m = -1$ is an eigenstate of L_z, hence one of our basis kets; in fact it is the third basis ket as we always list the eigenvalues in the order $\hbar$, 0, $-\hbar$. So

$$|m=-1\rangle \xrightarrow[L_z]{} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Per the question, write this as $|m=-1\rangle = a|E=0\rangle + |\phi\rangle$ where a and $|\phi\rangle$ are unknown, but $\langle\phi|E=0\rangle = 0$, i.e. $|\phi\rangle$ is a vector in the subspace with excited energy $E = A\hbar^2$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where

$$(1,0,-1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x-z) = 0,$$

so the excited subspace contains any vector with $x = z$. The solution may be obvious; formally we have to solve:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a/\sqrt{2}+x \\ y \\ -a/\sqrt{2}+x \end{pmatrix},$$

hence $y = 0$, $x = -a/\sqrt{2}$, and $-\sqrt{2}a = 1$:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The two terms have energies $A\hbar^2$ and 0, so we can apply the time evolution operator to get the state at time t:

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|m=-1\rangle = e^{-iA\hbar t}|\phi\rangle - e^0\frac{1}{\sqrt{2}}|E=0\rangle \xrightarrow[L_z]{} \frac{1}{2}\begin{pmatrix} e^{-iA\hbar t}-1 \\ 0 \\ e^{-iA\hbar t}+1 \end{pmatrix}.$$

This is a bit more illuminating if we pull out an overall phase factor:

$$|\Psi(t)\rangle \xrightarrow[L_z]{} \frac{e^{-iA\hbar t/2}}{2}\begin{pmatrix} e^{-iA\hbar t/2}-e^{iA\hbar t/2} \\ 0 \\ e^{-iA\hbar t/2}+e^{iA\hbar t/2} \end{pmatrix} = e^{-iA\hbar t/2}\begin{pmatrix} -i\sin(A\hbar t/2) \\ 0 \\ \cos(A\hbar t/2) \end{pmatrix}.$$

The ion oscillates between $L_z = -\hbar$ and $L_z = \hbar$, but without passing through $L_z = 0$.

5. First find the normalization constant N:

$$\langle\psi|\psi\rangle = 1 = |N|^2(1+3^2+2^2) = 14N^2$$

using the orthogonality of states with different eigenvalues. Hence $N^2 = 1/14$. The z component of angular momentum is given by the m quantum number, namely $L_z = m\hbar$, so the possibilities are: $L_z = 0$, probability 1/14; $L_z = -\hbar$, probability 9/14, $L_z = 2\hbar$, probability 4/14.