PHYS 20602 **Examples Class 2**

Finger Exercises (mostly from Q1 of the exam paper in various past years)

1. Give the number of dimensions of the following vector spaces:

   (i) The direct sum of a 2-D and a 3-D vector space: $V^2 \oplus V^3$;

   (ii) The set of polynomials of degree up to N;

2. For each of the following matrices, state whether it is Hermitian, unitary, both, or neither:

   (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (ii) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; (iii) $\begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$; (iv) $\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$.

   (From 2010 exam question; worth 6 marks)

3. Given that $\hat{\Omega}$ and $\hat{\Lambda}$ are Hermitian operators, and λ is a scalar, under what conditions are the following Hermitian or anti-Hermitian?

   NB: the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

   (a) $\hat{\Omega}\hat{\Lambda}$;

   (b) $\hat{\Omega}\hat{\Lambda} + \lambda\hat{\Omega}$;

   (c) $[\hat{\Omega}, \hat{\Lambda}]$;

   (d) $i[\hat{\Omega}, \hat{\Lambda}]$?

4. Show that the vectors
$$|\leftarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$
   and
$$|\rightarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$$
   are orthogonal, given that $|\uparrow\rangle$ and $|\downarrow\rangle$ are orthogonal.

5. A particle is in a state
$$|\psi(t=0)\rangle = C\left(|E_0\rangle + 2i|E_1\rangle + (1+i)|E_2\rangle\right)$$
   where $|E_0\rangle$, $|E_1\rangle$, $|E_2\rangle$ are energy eigenstates with energies $E_0 = 0.5$ eV, $E_1 = 1.5$ eV, and $E_2 = 2.5$ eV.

   (a) Evaluate the normalization constant C.

   (b) What are the possible results of a measurement of the energy, and the probability of each result?

(c) What is the expectation value of the energy?

(d) Write down the state $|\psi(t)\rangle$ of the particle at a later time t, assuming no measurement has taken place.

(e) Write down the state at time t assuming an ideal measurement took place and the energy was found to be E_0.

(Brownie points for anyone who can spot what kind of potential the particle is in, given the pattern of energy eigenvalues...you met this in Wendy Flavell's course!).

6. Find the eigenvectors and eigenvalues of the matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

(From 2009 exam paper, worth 7 marks).

Brownie points for guessing why this matrix is called "P"!

1. (i) 5 (dimensions of vector spaces just add with direct sums); (ii) Polynomials of degree N have the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_N x^N.$$

As we saw in an example, polynomials of the form x^n are linearly independent, so we need $N+1$ of them (including $x^0 = 1$ for the first term) to make any polynomial of degree N. Hence we have $N+1$ dimensions for the vector space describing these polynomials.

2. Matrix adjoints are:

(i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (ii) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; (iii) $\begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$; (iv) $\begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$.

Adjoints (i) and (iii) are equal to their original matrices, which are therefore Hermitian. Matrix multiplication of the adjoints with the originals gives the identity matrix for (i) and (iv), which are therefore unitary. (ii) is neither.

3. (a) $(\hat{\Omega}\hat{\Lambda})^\dagger = \hat{\Lambda}^\dagger\hat{\Omega}^\dagger = \hat{\Lambda}\hat{\Omega}$ since $\hat{\Lambda}$ and $\hat{\Omega}$ are Hermitian. Hence $\hat{\Omega}\hat{\Lambda}$ will be Hermitian if $[\hat{\Omega}, \hat{\Lambda}] = 0$, and anti-Hermitian if $\hat{\Omega}\hat{\Lambda} = -\hat{\Lambda}\hat{\Omega}$ (this is called anti-commutation). If neither of these holds, the product is neither Hermitian nor anti-Hermitian.

   (b) $(\hat{\Omega}\hat{\Lambda} + \lambda\hat{\Omega})^\dagger = \hat{\Lambda}\hat{\Omega} + \lambda^*\hat{\Omega}$. Hence the conditions are as in the previous part except that λ must be real for Hermitian and imaginary for anti-Hermitian.

   (c) $([\hat{\Omega}, \hat{\Lambda}])^\dagger = (\hat{\Omega}\hat{\Lambda} - \hat{\Lambda}\hat{\Omega})^\dagger = \hat{\Lambda}^\dagger\hat{\Omega}^\dagger - \hat{\Omega}^\dagger\hat{\Lambda}^\dagger = \hat{\Lambda}\hat{\Omega} - \hat{\Omega}\hat{\Lambda} = [\hat{\Lambda}, \hat{\Omega}] = -[\hat{\Omega}, \hat{\Lambda}]$. Therefore the commutator of Hermitian matrices is anti-Hermitian (or zero).

   (d) From the previous part, $i[\hat{\Omega}, \hat{\Lambda}]$ must be Hermitian.

4. If two kets are orthogonal their inner product is zero, i.e. $\langle \rightarrow | \leftarrow \rangle = 0$. We have

$$\begin{aligned} \langle \rightarrow | \leftarrow \rangle &= \left(\frac{\langle \uparrow | - \langle \downarrow |}{\sqrt{2}} \right) \left(\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \right) \\ &= \frac{\langle \uparrow | \uparrow \rangle + \langle \uparrow | \downarrow \rangle - \langle \downarrow | \uparrow \rangle - \langle \downarrow | \downarrow \rangle}{2} \\ &= \frac{\langle \uparrow | \uparrow \rangle + 0 - 0 - \langle \downarrow | \downarrow \rangle}{2} \end{aligned}$$

where we used orthogonality of $\{|\uparrow\rangle, |\downarrow\rangle\}$ in the last line. I should have told you also to assume that these vectors are normalised (this is the usual convention). If you assume that they are, then you get

$$\langle \rightarrow | \leftarrow \rangle = \frac{1 + 0 - 0 - 1}{2} = 0$$

as required.

(Similarly, $\langle \leftarrow | \leftarrow \rangle = 1 = \langle \rightarrow | \rightarrow \rangle$).

5. (a) Vectors representing physical states must be normalised, hence $\langle \psi | \psi \rangle = 1$. The states $|E_0\rangle$, $|E_1\rangle$ and $|E_2\rangle$ must be mutually orthogonal as they are eigenkets of a Hermitian operator (i.e. the Hamiltonian) with different eigenvalues. By the usual convention, as eigenkets, they are normalised. Hence we can ignore the cross terms in the inner product and we have:

$$\begin{aligned} 1 &= \langle \psi | \psi \rangle \\ &= C^* \left(\langle E_0 | - 2i \langle E_1 | + (1-i) \langle E_2 | \right) C \left(|E_0\rangle + 2i|E_1\rangle + (1+i)|E_2\rangle \right) \\ &= C^* C \left(\langle E_0 | E_0 \rangle - 2i \times 2i \langle E_1 | E_1 \rangle + (1-i)(1+i) \langle E_2 | E_2 \rangle \right) \\ &= |C|^2 (1 + 4 + 2) = 7|C|^2 \end{aligned}$$

Hence $C = e^{i\delta}/\sqrt{7}$ for some $\delta \in \mathbb{R}$.

(b) Possible results: E_0, E_1, E_2; Probabilities 1/7, 4/7, 2/7 respectively.

(c) $\langle E \rangle = 1\,\text{eV} \times \left(\frac{1}{2} \times \frac{1}{7} + \frac{3}{2} \times \frac{4}{7} + \frac{5}{2} \times \frac{2}{7} \right) = 23/14$ eV $= 1.64$ eV

(d)

$$|\psi(t)\rangle = C \left(e^{-iE_0 t/\hbar} |E_0\rangle + 2i e^{-iE_1 t/\hbar} |E_1\rangle + (1+i) e^{-iE_2 t/\hbar} |E_2\rangle \right)$$

(e)

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar} |E_0\rangle$$

For brownie points: A system with energies proportional to (1/2), (3/2), (5/2) ...times some basic unit sounds a lot like a 1-D simple harmonic oscillator.

6.

$$\det P = \begin{vmatrix} 1/2 - p & 1/2 \\ 1/2 & 1/2 - p \end{vmatrix} = \left(\frac{1}{2} - p \right)^2 - \left(\frac{1}{2} \right)^2 = \frac{1}{4} - p + p^2 - \frac{1}{4} = p(1-p).$$

so the eigenvalues (solutions of $\det P = 0$) are $p = 0$ and $p = 1$.

Eigenvectors:

$$\begin{pmatrix} 1/2 - p & 1/2 \\ 1/2 & 1/2 - p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

For $p = 0$ this gives $x/2 + y/2 = 0$ (twice), i.e. $x = -y$, or a normalized eigenvector of

$$|p = 0\rangle \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $p = 1$ we have $-x/2 + y/2 = 0$ and (redundantly, as ever when solving for eigenvectors) $x/2 - y/2 = 0$, i.e. $x = y$, or

$$|p = 1\rangle \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\hat{P}$ is an operator with a single non-zero eigenvalue, and that equal to 1. That makes it a projector (onto the state $|p = 1\rangle$).